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Abstract. We describe a method to show short time uniqueness results for viscosity solutions of general nonlocal and non-monotone second-order geometric equations arising in front propagation problems. Our method is based on some lower gradient bounds for the solution. These estimates are crucial to obtain regularity properties of the front, which allow to deal with nonlocal terms in the equations. Applications to short time uniqueness results for the initial value problems for dislocation type equations, asymptotic equations of a FitzHugh-Nagumo type system and equations depending on the Lebesgue measure of the fronts are presented.

1. Introduction

We are concerned with the evolution of compact hypersurfaces $\{\Gamma_t\}_{t\geq 0} \subset \mathbb{R}^N$ moving according to the general non-local law of propagation

$$V = h(x, t, \Omega_t, n(x), Dn(x)) \quad \text{on } \Gamma_t, \tag{1}$$

where V is the normal velocity of Γ_t which depends, through the evolution law h, on time, on the position of $x \in \Gamma_t$, on the set Ω_t enclosed by Γ_t, on the unit normal $n(x)$ to Γ_t at x pointing outward to Ω_t and on its gradient $Dn(x)$ which carries the curvature dependence of the velocity.

When such motion is local, i.e., when h does not depend on Ω_t, and satisfies *the inclusion principle* or *geometrical monotonicity*, i.e., when, at least formally, the inclusion $\Omega_0^1 \subset \Omega_0^2$ at time $t = 0$ implies $\Omega_t^1 \subset \Omega_t^2$ for any $t > 0$, it is proved by Souganidis and the first author [12] that the motion can be defined and studied by the *level set approach*, which was introduced by Osher and Sethian [40] for numerical calculations and then developed, from a theoretical point of view, by Evans and Spruck [25] for the mean curvature motion and by Chen, Giga and Goto [19] for general velocities. This approach replaces the geometrical problem (1) with a degenerate parabolic partial differential equation called the *geometric* or *level set*

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equation. This equation is designed to describe the desired evolution via the 0-level set of its solution. More precisely, the existence and uniqueness of the level set solution $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ allows to define Γ_t as being the set $\{x \in \mathbb{R}^N \mid u(\cdot, t) = 0\}$.

In recent years, there has been much interest on the study of front propagations problems in cases when the normal velocity of the front depends on a non-local way of the enclosed region like (1). This interest was motivated by several types of applications like dislocations' theory or FitzHugh-Nagumo type systems or volume dependent velocities that we describe below. It is worth pointing out that the level set approach still applies for motions with nonlocal velocities provided that the inclusion principle holds, following the ideas of Slepcev [42]. But, in many of the above mentioned applications, one faces non-monotone surface evolution equations. For such class of problems, the level set approach cannot be used directly since the classical comparison arguments of viscosity solutions' theory fail and therefore, the existence and uniqueness of viscosity solutions to these equations become an issue.

Though the existence properties for such motions seem now to be well understood (see [30, 43, 8]), this is not the case for uniqueness. In particular, there are not many uniqueness results for curvature dependent velocities. As far as the authors know, there are only two works by Forcadel [27] and Forcadel and Monteillet [28] which investigate the motion arising in a model for dislocation dynamics which is included by our general equations. The aim of this article is to consider cases where we have, at the same time, a non-local velocity which induces a non-monotone evolution together with a curvature dependence (we explain later on the state of the art for such problems and why the curvature dependence creates a specific difficulty). More specifically, we describe a method to show short time uniqueness results for the general motion (1).

We now describe some typical applications we have in mind. We first consider a model for dislocation dynamics

$$h = M(n(x))\big(c_0(\cdot, t) * \mathbf{1}_{\overline{\Omega}_t}(x) + c_1(x, t) - \operatorname{div}_{\Gamma_t}\xi(n(x))\big), \tag{2}$$

where $\mathbf{1}_A$ denotes the indicator function of a subset A of $\mathbb{R}^N$ and $M, \xi : \mathbb{S}^{N-1} \to \mathbb{R}$, $c_0, c_1 : \mathbb{R}^N \times [0, T] \to \mathbb{R}$ are given functions and we write

$$c_0(\cdot, t) * \mathbf{1}_{\overline{\Omega}_t}(x) := \int_{\mathbb{R}^N} c_0(x - y, t)\mathbf{1}_{\overline{\Omega}_t}(y)\, dy.$$

Here, $\mathbb{S}^{N-1}$ denotes the $(N-1)$-dimensional unit sphere. The term $\operatorname{div}_{\Gamma_t}\xi(n(x)) := \operatorname{tr}\big((I - n(x) \otimes n(x))D_x\xi(n(x))\big)$ is called the *anisotropic* (or *weighted*) *mean curvature* of Γ_t at x (in the direction of $n(x)$). See for instance Giga [29]. Typically, the reasonable assumptions in this context are the following: M is a positive and bounded function, c_0, c_1 are bounded, continuous functions which are Lipschitz continuous in x variable (uniformly with respect to t variable), $c_0, D_xc_0 \in L^\infty([0, T]; L^1(\mathbb{R}^N))$ and ξ is a positively homogeneous function with degree 0. The surface evolution equation (2) without the last term in the right hand side is well-known as typical models of the dislocation dynamics (see [41, 2] for a derivation and the physical background).

We next consider asymptotic equations of a FitzHugh-Nagumo type system as an example of interface dynamics coupled with a diffusion equations,

$$
\begin{gathered}
h = \alpha(v(x,t)) - \operatorname{div}_{\Gamma_t}(n(x)) \\
\text{and} \\
v_t - \Delta v = g^+(v)\mathbf{1}_{\overline{\Omega}_t} + g^-(v)(1 - \mathbf{1}_{\overline{\Omega}_t}),
\end{gathered} \tag{3}
$$

where $\alpha, g^{\pm} : \mathbb{R} \to \mathbb{R}$ are bounded and Lipschitz continuous with $g^- \leq g^+$. This system has been investigated by Giga, Goto and Ishii [30] and Soravia and Souganidis [43].

Finally, we consider equations depending on the measure of the fronts like

$$
h = \beta(\mathcal{L}^N(\overline{\Omega}_t)) - \operatorname{div}_{\Gamma_t}(n(x)), \tag{4}
$$

where the function $\beta : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. A typical example is $\beta(r) = a + br$ for some $a, b \in \mathbb{R}$ which has been investigated by Chen, Hilhorst and Logak in [20] (see also [17, 18]).

As we already mentioned it above, these examples are not only nonlocal but also non-monotone surface evolution equations. Indeed, in (2), the kernel c_0 may change sign and, in (3) and in (4), the functions α, β may be non-monotone. We also refer to [17, 42, 23, 44] for some monotone non-local geometric equations. By using the framework which we present in this paper, we give short time uniqueness results for (2), (3) and (4).

There are many results of existence and uniqueness for the simplest case of motions of (2), i.e., $M(p) \equiv 1$ and $\xi(p) = p/|p|$ without a curvature term. A short time existence and uniqueness result was first obtained in [2]. But then most of the results were obtained for curvature-independent velocities ($\xi = 0$): long time existence and uniqueness results were obtained when the velocity is positive, i.e.,

$$
h > 0 \quad \text{on } \Gamma_t, \tag{5}
$$

by Alvarez, Cardaliaguet and Monneau in [1] and by the first two authors in [10] by different methods. The first two authors with Cardaliaguet and Monneau in [6] presented a new notion of weak solutions (see Definition 1) of the level set equation for (2) without a curvature term, gave the global existence of these weak solutions and analysed the uniqueness of them when (5) holds. A similar concept of solutions already appeared in [30, 43]. In the companion paper [7], the first two authors with Cardaliaguet and Monteillet proposed a new perimeter estimate for the evolving fronts with uniform interior cone property and by using this, they extended the uniqueness result for dislocation dynamics equations and provided the uniqueness result for asymptotic equations of a FitzHugh-Nagumo type system, still under the positiveness assumption (5).

In this paper, we do not use the perimeter estimate in an essential way but either elementary measure estimates or, in the most sophisticated cases, the interior cone property (see Lemma 11). Since the studies by [1, 10, 6, 7], it is now well-known that estimates on *lower gradient bound* and perimeter of 0-level sets of viscosity solutions of associated local equations are key properties to obtain existence and uniqueness results for nonlocal equations derived from (2), (3) and (4). Let us describe the main difficulty of our problem and, to do so, we consider the level set equations of the simplest case of (2) or (3) here. Considering the non-local part as a given function, we are led to the study the (local) initial value problem

$$\begin{cases} u_t = \Big(c(x,t) + \mathrm{div}\,\big(\dfrac{Du}{|Du|}\big)\Big)|Du| & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{6}$$

where $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ and $c \in C(\mathbb{R}^N \times [0,T])$ are bounded and Lipschitz continuous with respect to the x variable. One of our main results is a short time lower gradient bound estimate for the viscosity solution of (6), i.e.,

$$|Du(x,t)| \geq \eta(t) > 0 \quad \text{in a neighborhood of } \{u(\cdot,t) = 0\}. \tag{7}$$

For first-order eikonal equations, lower gradient bound comes naturally from the Barron-Jensen's approach (see [37]). For second-order equations like (6), it is affected by the "*diffusion*" term and the *non-empty interior* difficulty and therefore we cannot expect that the property (7) holds generally and for long-time. Indeed, in [13], see also [36, 31], they consider the simple example of (6) with $c \equiv 1$ and smooth u_0 such that $Du_0 \neq 0$ on the initial front $\{u_0 = 0\}$. They prove that, up to choose suitable u_0, fattening may occur for arbitrary $t > 0$, i.e., the front may develop an interior. It is precisely this reason which implies that there are not many results on the nonlocal second-order equations like the level sets equations of (2), (3) and (4) and a short-time result is optimal.

Existence results were obtained by [30, 43, 8] but they concern merely existence of weak solutions defined by Definition 1. As stated above, there are two works [27, 28] which give uniqueness results for the motion (2). The difference between our results and theirs is that, in [27], only the evolution of hypersurfaces which can be expressed by graphs of functions is considered while, in [28], the arguments are based on minimizing movement for (2) and they are completely different from our arguments which are based on the theory of viscosity solutions. Moreover, for the existence of minimizing movement for the simplest case of (2), the assumptions that $c_0(\cdot,t)$ is symmetry and c_0, c_1 are smooth enough are essentially used. Therefore, uniqueness results for the examples (3) and (4) are not covered by [28].

Another difference with existing results in the literature is that h is allowed to change sign in (1), contrary to [1, 10, 6, 7] where (5) is one of the main assumption to get uniqueness. It may give rise of fattening, see [11, Proposition 4.4], and it is another explanation of the short time result.

Finally, we explain the key idea to obtain (7) for viscosity solutions of (6). In order to get it, we make the following assumption on u_0. There exist constants

$\lambda_0 \in (0,1)$, $\eta_0 > 0$ and $\nu \in C(\mathbb{R}^N, \mathbb{R}^N)$ such that

$$u_0(x + \lambda\nu(x)) \geq u_0(x) + \lambda\eta_0 \text{ in a neighborhood of } \{u_0(\cdot) = 0\}$$

for all $\lambda \in [0, \lambda_0]$. Then we prove that such a property is preserved for the solution of (6), at least for short time, i.e.,

$$u(x + \lambda\nu(x), t) \geq u(x,t) + \lambda\eta(t) \text{ in a neighborhood of } \{u(\cdot, t) = 0\} \tag{8}$$

for all $\lambda \in [0, \overline{\lambda}]$, $t \in [0, \overline{t} \wedge T]$ and some $\overline{t} > 0$, $\overline{\lambda} \in (0, \lambda_0]$, where $\eta : [0, \overline{t} \wedge T] \to [0, \infty)$ is a non-increasing continuous function such that

$$\eta(t) > 0 \text{ for all } t \in [0, \overline{t} \wedge T). \tag{9}$$

The assumption on u_0 is inspired by [11, Theorem 4.3], where it is formulated only for the sign-distance function. A similar result to (8) may be found in [14] where it is used to prove uniqueness results for the mean curvature motion for entire graphs.

If u_0 is a smooth function with $Du_0 \neq 0$ on the compact hypersurface $\Gamma_0 = \{x : u_0(x) = 0\}$, then the assumption is satisfied with $\nu(x) = Du_0(x)$ and if there exists a smooth solution of the level set equation, then (8) holds for short time. But, on one hand, the general degenerate parabolic and nonlinear equations we consider do not have classical solutions in general, and on the other hand, the above assumption on u_0 is valid in cases when Γ_0 is not a smooth hypersurface, which is also an important point here.

The proof of (8) uses in a crucial way the geometric property of (6) and a *continuous dependence result* for parabolic problems (which is, by the way, of independent interest). We refer to [34, 35, 9] and references therein for the detail of the continuous dependence result for elliptic and parabolic problems.

We derive lower gradient estimate (7) from (8) formally here. We have

$$\begin{aligned}
\lambda\eta(t_0) &\leq u(x_0 + \lambda\nu(x_0), t_0) - u(x_0, t_0) \\
&= \lambda\langle Du(x_0, t_0), \nu(x)\rangle + o(\lambda\|\nu\|_\infty) \\
&\leq \lambda|Du(x_0, t_0)|\|\nu\|_\infty + o(\lambda\|\nu\|_\infty) \quad \text{in a neighborhood of } \{u(\cdot, t) = 0\}
\end{aligned}$$

for all $t \in [0, \overline{t} \wedge T]$ with $o(r)/r \to 0$ as $r \to 0$. Dividing λ in the above and taking a sufficiently small $\lambda \in (0, \overline{\lambda}]$, we get the lower estimate (7). We also obtain the interior cone property of fronts by (8).

The paper is organized as follows: in Section 2, we state a continuous dependence result for a class of equations which encompasses level set equations associated to (1). In Section 3, we obtain the key estimate (8) and derive the lower-bound gradient and perimeter estimates of 0-level sets of viscosity solutions of local equations. In Section 4, we consider the *level set* equation of (1) and give the proof for the short time uniqueness result (Theorem 10). Section 5 is devoted to existence and uniqueness results for the level set equations of (2), (3) and (4) as applications of Theorem 10.

Notations. For some $k \in \mathbb{N}$, we denote by $\mathbb{R}^k$ the k-dimensional Euclidean space equipped with the usual Euclidean inner product $\langle \cdot, \cdot \rangle$, and by $\mathcal{S}^k$ the space of $k \times k$

symmetric matrices. We write $B(x,r)=\{y\in\mathbb{R}^k \mid |x-y|<r\}$ for $x\in\mathbb{R}^k$, $r>0$, and $A+rB(0,1):=\{x+y \mid x\in A, y\in B(0,r)\}$ for $A\subset\mathbb{R}^k$. The symbols $\mathcal{L}^k(A)$ and $\mathcal{H}^k(A)$ denote the k-dimensional Lebesgue and Hausdorff measures, respectively. We write X^T for the transpose of the matrix X and $|X|=\sup\{|X\xi| \mid \xi\in\mathbb{R}^k, |\xi|=1\}$. Finally, for $a,b\in\mathbb{R}$, we write $a\wedge b=\min\{a,b\}$ and $a\vee b=\max\{a,b\}$.

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2. Continuous Dependence of Solutions

In this section, we are concerned with the equation

$$u_t+H(x,t,Du,D^2u)=0 \quad \text{in } \mathbb{R}^N\times(0,T), \tag{10}$$

$T>0$, $u:\mathbb{R}^N\times(0,T)\to\mathbb{R}$ is the unknown function, u_t, Du and D^2u stand respectively for its time and space derivatives, and Hessian matrix with respect to x variable. We use the following assumptions.

(A1) $H\in C(\mathbb{R}^N\times[0,T]\times(\mathbb{R}^N\setminus\{0\})\times\mathcal{S}^N)$.

(A2) The equation is *degenerate parabolic*, i.e.,

$$H(x,t,p,X)\geq H(x,t,p,Y),$$

for any $(x,t,p)\in\mathbb{R}^N\times[0,T]\times(\mathbb{R}^N\setminus\{0\})$ and $X,Y\in\mathcal{S}^N$ with $X\leq Y$, where $\leq$ stands for the usual partial ordering for symmetric matrices.

(A3) For any $(x,t)\in\mathbb{R}^N\times[0,T]$, $H^*(x,t,0,0)=H_*(x,t,0,0)$, where H^* (resp., H_*) is the upper-semicontinuous envelope (resp., lower semicontinuous envelope) of H.

(A4) There exist $\kappa_1,\kappa_2\geq 0$, $M\geq 0$ such that

$$\begin{aligned} &H_2(y,t,p,Y)-H_1(x,t,p,X)\\ &\leq C_R\Big(\frac{|x-y|^4}{\varepsilon^4}+\kappa_1+\frac{\kappa_2|x-y|^2}{\varepsilon^4}+\rho\|A^2\|\Big) \end{aligned} \tag{11}$$

for any $\rho,\varepsilon\in(0,1)$, $R>0$, $x,y\in\overline{B}(0,R)$, $p=4\varepsilon^{-4}|x-y|^2(x-y)$, $X,Y\in\mathcal{S}^N$ and some $C_R>0$ satisfying

$$\begin{cases} |p|\leq M,\\ \begin{pmatrix} X & 0\\ 0 & -Y\end{pmatrix}\leq A+\rho A^2, \end{cases} \tag{12}$$

where

$$A:=\frac{|x-y|^2}{\varepsilon^4}\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}+\frac{|x-y|^2}{\varepsilon^4}\begin{pmatrix} \hat p\otimes\hat p & -\hat p\otimes\hat p\\ -\hat p\otimes\hat p & \hat p\otimes\hat p\end{pmatrix}, \tag{13}$$

with $\hat p:=p/|p|$.

We note that, in this section, we do not assume that H is geometric.

Theorem 1. *Let H_1, H_2 be functions on $\mathbb{R}^N \times [0,T] \times (\mathbb{R}^N \setminus \{0\}) \times \mathcal{S}^N$ satisfying assumptions* (A1)–(A4). *Let $u_1, u_2 \in C(\mathbb{R}^N \times [0,T])$ be, respectively, a bounded viscosity subsolution and viscosity supersolution of* (10) *with $H = H_i$ for $i = 1, 2$. Assume that there exists $L = L_{u_i} > 0$ such that*

$$|u_i(x,t) - u_i(y,t)| \le L|x-y| \quad \text{for all } x, y \in \mathbb{R}^N,\ t \in [0,T] \tag{14}$$

for either $i = 1$ or 2, and that there exists $R > 0$ such that

$$u_i(x,t) = -1 \quad \text{for all } x \in \mathbb{R}^N \setminus B(0,R),\ t \in [0,T] \tag{15}$$

for both $i = 1$ and 2. Then there exists $M_1 > 0$ which depends only on C, L, $\|u_1\|_\infty$ and $\|u_2\|_\infty$ such that

$$\sup_{x\in\mathbb{R}^N} (u_1 - u_2)(x,t) \le \sup_{x\in\mathbb{R}^N} (u_1 - u_2)(x,0) + M_1\big(\kappa_1 t + (\kappa_2 t)^{1/2}\big) \tag{16}$$

for all $t \in [0,T]$.

Remark 1. An assumption like (A4) is natural in viscosity theory to obtain continuous dependence results of the type (16) and the regularity of the solution (cf. (14)) is a key ingredient too, see [9, 34, 35]. In Example 1 below, we show that (A4) holds in the cases we are interested in. Note that (15) are not restrictive assumptions when dealing with front propagation problems, see [29, 10, 7, 8].

Proof. Let $\varepsilon \in (0,1)$ and $K > 0$. We shall later fix ε, K. Consider

$$\sup_{x,y\in\mathbb{R}^N, t\in[0,T]} \{u_1(x,t) - u_2(y,t) - \frac{|x-y|^4}{\varepsilon^4} - Kt\}.$$

Noting (15), it is clear that the supremum is attained at $(\overline{x}, \overline{y}, \overline{t}) \in \overline{B}(0, R+1)^2 \times [0,T]$ for small $\varepsilon > 0$.

We consider the case where $\overline{t} \in (0,T]$. In view of Ishii's Lemma, for any $\rho > 0$, there exist $(a, p, X) \in \overline{\mathcal{J}}^{2,+} u_1(\overline{x}, \overline{t})$ and $(b, p, Y) \in \overline{\mathcal{J}}^{2,-} u_2(\overline{y}, \overline{t})$ (see [22] for the notation) such that

$$a - b \ge K, \quad p = \frac{4|\overline{x}-\overline{y}|^2}{\varepsilon^4}(\overline{x}-\overline{y}), \quad \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \rho A^2, \tag{17}$$

where A is the matrix defined by (13). The definition of viscosity solutions immediately implies the following inequalities:

$$a + (H_1)_*(\overline{x}, \overline{t}, p, X) \le 0, \quad b + (H_2)^*(\overline{y}, \overline{t}, p, Y) \ge 0.$$

Hence we have

$$K + (H_1)_*(\overline{x}, \overline{t}, p, X) - (H_2)^*(\overline{y}, \overline{t}, p, Y) \le 0. \tag{18}$$

Using that u_1 or u_2 is Lipschitz continuous with respect to x variable, we get, by standard estimates,

$$|\overline{x} - \overline{y}| \le M\varepsilon^{4/3}, \quad |p| \le M,$$

where M is a positive constant which depends only on $\|u_1\|_{L^\infty(\mathbb{R}^N\times[0,T])}$, $\|u_2\|_{L^\infty(\mathbb{R}^N\times[0,T])}$ and L.

We now distinguish two cases: (i) for any $\varepsilon \in (0,1)$, $p \neq 0$; (ii) there exist $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that $\varepsilon_j \to 0$ as $j \to \infty$, $p = 0$ for any $j \in \mathbb{N}$.

We first consider case (i). In view of (A4), we have

$$\begin{aligned}K \leq& H_2(\overline{y},\overline{t},p,Y) - H_1(\overline{x},\overline{t},p,X)\\ \leq& C_R\big(\frac{|\overline{x}-\overline{y}|^4}{\varepsilon^4} + \kappa_1 + \frac{\kappa_2|\overline{x}-\overline{y}|^2}{\varepsilon^4} + \rho\|A^2\|\big).\end{aligned}$$

Sending $\rho \to 0$, we get

$$\begin{aligned}K \leq& C_R\big(\frac{|\overline{x}-\overline{y}|^4}{\varepsilon^4} + \kappa_1 + \frac{\kappa_2|\overline{x}-\overline{y}|^2}{\varepsilon^4}\big)\\ \leq& \tilde{C}\big(\varepsilon^{4/3} + \kappa_1 + \frac{\kappa_2}{\varepsilon^{4/3}}\big) =: C_\varepsilon\end{aligned}$$

In case (ii), we have $\overline{x} = \overline{y}$. Due to (17), we have $A = 0$, $X \leq 0$ and $Y \geq 0$. By (A2), we have

$$(H_1)_*(\overline{x},\overline{t},0,X) \geq (H_1)_*(\overline{x},\overline{t},0,0) \text{ and } (H_2)^*(\overline{y},\overline{t},0,Y) \leq (H_2)^*(\overline{y},\overline{t},0,0).$$

Therefore, we get

$$\begin{aligned}K \leq& (H_2)^*(\overline{y},\overline{t},0,Y) - (H_1)_*(\overline{x},\overline{t},0,X)\\ \leq& (H_2)^*(\overline{y},\overline{t},0,0) - (H_1)_*(\overline{x},\overline{t},0,0) = 0.\end{aligned}$$

Set $K = C_\varepsilon + \tilde{C}\varepsilon^{4/3}$ and then the two above cases cannot hold; this means that necessarily we have $\overline{t} = 0$. Therefore, for any $(x,t) \in \overline{B}(0,R) \times [0,T]$,

$$\begin{aligned}&(u_1 - u_2)(x,t)\\ \leq& u_1(\overline{x},0) - u_2(\overline{y},0) + Kt\\ \leq& \sup_{x\in\mathbb{R}^N}(u_1-u_2)(x,0) + ML\varepsilon^{4/3} + \tilde{C}(2\varepsilon^{4/3} + \kappa_1 + \frac{\kappa_2}{\varepsilon^{4/3}})t.\end{aligned}$$

An optimization with respect to $\varepsilon > 0$ yields

$$(u_1 - u_2)(x,t) \leq \sup_{x\in\mathbb{R}^N}(u_1-u_2)(\cdot,0) + M_1\big(\kappa_1 t + (\kappa_2 t)^{1/2}\big)$$

for some $M_1 = M_1(C_R, L, \|u_1\|_\infty, \|u_2\|_\infty) > 0$. □

Example 1. We consider the functions $H_i : \mathbb{R}^N \times [0,T] \times (\mathbb{R}^N \setminus \{0\}) \times \mathcal{S}^N$ defined by

$$H_i(x,t,p,X) = \inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\big\{-c_i^{\alpha,\beta}(x,t,p)|p| - \mathrm{tr}\,\big(\sigma_i^{\alpha,\beta}(x,t,p)(\sigma_i^{\alpha,\beta})^T(x,t,p)X\big)\big\} \quad (19)$$

for $i = 1,2$, where $\mathcal{A},\mathcal{B}$ are compact metric space and $c_i^{\alpha,\beta}$, $\sigma_i^{\alpha,\beta}$ are, respectively, real-valued functions and $m \times N$ matrix valued functions for some $m \in \mathbb{N}$ on $\mathbb{R}^N \times [0,T] \times (\mathbb{R}^N \setminus \{0\})$ with a possible singularity at $p = 0$. We assume that the functions $c_i^{\alpha,\beta}$, $\sigma_i^{\alpha,\beta}$ satisfy the following conditions by replacing h by $c_i^{\alpha,\beta}$, $\sigma_i^{\alpha,\beta}$ for any $\alpha \in \mathcal{A}$,

$\beta \in \mathcal{B}$, $i = 1, 2$, respectively: h are continuous on $\mathbb{R}^N \times [0, T]$ and for some $L, M \geq 0$ (independent of α, β),

$$\begin{gathered} |h(x,t,p) - h(y,t,p)| \leq L|x-y|, \\ |h(x,t,p)| \leq M \end{gathered} \tag{20}$$

for all $x, y \in \mathbb{R}^N$, $t \in [0, T]$, $p \in \mathbb{R}^N \setminus \{0\}$.

Let $\rho, \varepsilon \in (0, 1)$, $x, y \in \overline{B}(0, R)$, $p = \varepsilon^{-4}|x-y|^2(x-y)$, $X, Y \in \mathcal{S}^N$ satisfy (12) for some $R > 0$ and let A be the matrix given by (13). We omit the dependence of α, β for simplicity of notation. We calculate that

$$\begin{aligned} &(c_1(x,t,p) - c_2(y,t,p))|p| \\ &= (c_1(x,t,p) - c_1(y,t,p))|p| + (c_1(y,t,p) - c_2(y,t,p))|p| \\ &\leq |p|(L|x-y| + \|c_1 - c_2\|_\infty) \\ &\leq L\frac{|x-y|^4}{\varepsilon^4} + M\|c_1 - c_2\|_\infty, \end{aligned}$$

where $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(B(0,R)\times(0,T)\times(\mathbb{R}^N\setminus\{0\}))}$ and

$$\begin{aligned} &\operatorname{tr}(\sigma_1^x(\sigma_1^x)^T X) - \operatorname{tr}(\sigma_2^y(\sigma_2^y)^T Y) \\ &= \sum_{i=1}^N \big\{ \langle X\sigma_1^x e_i, \sigma_1^x e_i\rangle - \langle Y\sigma_2^y e_i, \sigma_2^y e_i\rangle \big\} \\ &\leq \sum_{i=1}^N \Big\{ \Big\langle A\begin{pmatrix} \sigma_1^x e_i \\ \sigma_2^y e_i \end{pmatrix}, \begin{pmatrix} \sigma_1^x e_i \\ \sigma_2^y e_i \end{pmatrix} \Big\rangle + \rho\Big\langle A^2\begin{pmatrix} \sigma_1^x e_i \\ \sigma_2^y e_i \end{pmatrix}, \begin{pmatrix} \sigma_1^x e_i \\ \sigma_2^y e_i \end{pmatrix} \Big\rangle \Big\} \\ &\leq \sum_{i=1}^N \Big\{ \Big\langle A\begin{pmatrix} \sigma_1^x e_i \\ \sigma_2^y e_i \end{pmatrix}, \begin{pmatrix} \sigma_1^x e_i \\ \sigma_2^y e_i \end{pmatrix} \Big\rangle + C\rho\|A^2\|(\|\sigma_1^x\|_\infty^2 + \|\sigma_2^x\|_\infty^2) \end{aligned}$$

for some $C > 0$, where $\{e_i\}_i$ is the canonical basis of $\mathbb{R}^N$, $\sigma_1^x := \sigma_1^{\alpha,\beta}(x,t,p)$ and $\sigma_2^y := \sigma_2^{\alpha,\beta}(y,t,p)$. Due to (12), we have

$$\begin{aligned} &\Big\langle A\begin{pmatrix} \sigma_1^x e_i \\ \sigma_2^y e_i \end{pmatrix}, \begin{pmatrix} \sigma_1^x e_i \\ \sigma_2^y e_i \end{pmatrix} \Big\rangle \leq \frac{2}{\varepsilon^4}|x-y|^2|(\sigma_1^x - \sigma_2^y)e_i|^2 \\ &\leq \frac{4}{\varepsilon^4}|x-y|^2\big\{|\sigma_1(x,t,p) - \sigma_1(y,t,p)|^2 + \|\sigma_1 - \sigma_2\|_\infty^2\big\} \\ &\leq \frac{4}{\varepsilon^4}|x-y|^2\big(L^2|x-y|^2 + \|\sigma_1 - \sigma_2\|_\infty^2\big). \end{aligned}$$

From the above computations, it follows that the inequality (A4) holds by replacing κ_1 and κ_2 by $\sup_{\mathcal{A}\times\mathcal{B}}\|c_1 - c_2\|_\infty$ and $\sup_{\mathcal{A}\times\mathcal{B}}\|\sigma_1 - \sigma_2\|_\infty^2$, respectively. Therefore, if the u_i's are solutions of (10) with $H = H_i$ given by (19), $i = 1, 2$, then the conclusion of Theorem 1 holds and reads

$$\begin{aligned} \sup_{x\in\mathbb{R}^N}(u_1 - u_2)(x,t) &\leq \sup_{x\in\mathbb{R}^N}(u_1 - u_2)(x,0) \\ &\qquad + M_1\sup_{\mathcal{A}\times\mathcal{B}}\big(t\|c_1^{\alpha,\beta} - c_2^{\alpha,\beta}\|_\infty + \sqrt{t}\|\sigma_1^{\alpha,\beta} - \sigma_2^{\alpha,\beta}\|_\infty\big) \end{aligned}$$

for all $t \in [0,T]$. Finally, note that, applying Theorem 1 with $H_1 = H_2$, gives comparison and uniqueness for (10).

Remark 2. For the applications we have in mind, a continuous dependence result for equations with a measurable dependence in time will be needed. We do not state a precise result here but we mention that it can be obtained by an easy approximation argument.

3. Estimates on Lower-Bound Gradient and Properties of Fronts

We consider the initial value problem in this section

$$\begin{cases} u_t + H(x,t,Du,D^2u) = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{21}$$

We make the following assumption on u_0 throughout this section.

(I1) $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ and $|u_0(x)| \le 1$ for all $x \in \mathbb{R}^N$ and there exists $R_0 > 0$ such that $u_0(x) = -1$ for all $x \in \mathbb{R}^N \setminus B(0,R_0)$.

(I2) There exist constants $\lambda_0, \delta_0 \in (0,1)$, $\eta_0 > 0$ and $\nu \in C(\mathbb{R}^N, \mathbb{R}^N)$ such that

$$u_0(x + \lambda\nu(x)) \ge u_0(x) + \lambda\eta_0 \quad \text{for all } x \in U_0,\ \lambda \in [0,\lambda_0], \tag{22}$$

where $U_0 := \{x \in \mathbb{R}^N \mid |u_0(x)| \le \delta_0\}$.

Remark 3. Without loss of generality, we may assume that ν is a smooth bounded Lipschitz continuous function and henceforth we will assume it from now on. Indeed, let $\nu_\varepsilon \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ for $\varepsilon \in (0,1)$ be an approximate function of ν, then we have

$$\begin{aligned} u_0(x + \lambda\nu_\varepsilon(x)) =&\ u_0(x + \lambda\nu(x) + \lambda(\nu_\varepsilon(x) - \nu(x))) \\ \ge&\ u_0(x + \lambda\nu(x)) - \lambda\|Du_0\|_{L^\infty(U_0)}\|\nu_\varepsilon - \nu\|_{L^\infty(U_0)}. \end{aligned}$$

If ε is enough small, then we have

$$u_0(x + \lambda\nu_\varepsilon(x)) \ge u_0(x) + \lambda\frac{\eta_0}{2} \quad \text{for all } x \in U_0,\ \lambda \in [0,\lambda_0].$$

Let $u_0 \in C^1(\mathbb{R}^N)$ such that

$$Du_0 \ne 0 \quad \text{on} \quad \Gamma_0 := \{u_0 = 0\}. \tag{23}$$

Then, for $\delta_0 > 0$ and λ_0 enough small, $Du_0 \ne 0$ in $U_0 + B(0,\lambda_0\|\nu\|_\infty)$ and, setting $\nu(x) = Du_0(x)$, we have

$$u_0(x + \lambda\nu(x)) = u_0(x) + \lambda|Du_0(x)|^2 + \lambda\omega_{U_0}(M\lambda),$$

where ω_{U_0} is a modulus of continuity of Du_0 in $\overline{U}_0 + B(0,\lambda_0\|\nu\|_\infty)$ and $M = \max_{\overline{U}_0+B(0,\lambda_0\|\nu\|_\infty)}|Du_0|$. Therefore (I2) holds for $\eta = \min_{\overline{U}_0+B(0,\lambda_0\|\nu\|_\infty)}|Du_0|/2$ and $\lambda \in [0,\lambda_0]$ for λ_0 enough small. Moreover, the Implicit Function Theorem implies that Γ_0 is a C^1 hypersurface. Conversely, assume that Γ_0 is a C^1 hypersurface with the *unique nearest point property* (that is, there exists a neighborhood U_0 of Γ_0 such that, for all $x \in U_0$, there exists a unique $\overline{x} \in \Gamma_0$ such that $\mathrm{dist}(x,\Gamma_0) = |x - \overline{x}|$). Then the signed distance function $d^s_{\Gamma_0}$ to Γ_0 is C^1 (see [26]). It follows that (I1), (I2) hold with u_0 such that $u_0 = d^s_{\Gamma_0}$ in a neighborhood of Γ_0 and u_0 is a suitable regularization of $d^s_{\Gamma_0}$ elsewhere. More generally, when considering front propagation problems, it may be convenient to have a characterization of (I2) in geometrical terms. Such a result does not seem obvious. However, we have partial results in the following lemma, the proof of which is given in the appendix with additional comments.

A subset $A \subset \mathbb{R}^N$ is *star-shaped* with respect to x_0 if, for every $x \in A$, the segment $[x_0, x) := \{\lambda x + (1-\lambda)x_0, \lambda \in [0,1)\}$ belongs to A. It is *star-shaped with respect to a ball* $B(x_0, r_0)$ if A is star-shaped with respect to every $y \in B(x_0, r_0)$.

Lemma 2. *Let $\Omega_0 \subset \mathbb{R}^N$ be an open bounded set with boundary $\partial\Omega_0 =: \Gamma_0$.*

(i) (Star-shaped with respect to a ball domains) *The set Ω_0 is star-shaped with respect to a ball, i.e., there exists a compact subset $\mathcal{K} \subset \mathbb{R}^N$ and $r_0 > 0$ such that*
$$\Omega_0 = \bigcup_{x\in\mathcal{K}} \bigcup_{\alpha\in[0,1]} \overline{B}(\alpha x, (1-\alpha)r_0), \tag{24}$$
if and only if there exists $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that
$$\Gamma_0 = \{u_0 = 0\}, \qquad \Omega_0 = \{u_0 > 0\} \tag{25}$$
and (I1), (I2) *hold in U_0 with $\nu(x) = -x$. In this case, Γ_0 is locally the graph of a Lipschitz continuous function.*

(ii) *If there exists $K > 0$ such that Γ_0 is locally the graph of a Lipschitz continuous function with constant K, then there exists u_0 such that* (25), (I1) *and* (I2) *hold.*

Hereinafter, we set
$$\psi_\lambda(x) := x + \lambda\nu(x) = (I + \lambda\nu)(x). \tag{26}$$
From Remark 3, we may assume that ν is a smooth bounded Lipschitz continuous function and, replacing λ_0 by a smaller constant in order that
$$\lambda_0\|\nu\|_\infty < 1 \quad \text{and} \quad \lambda_0\|D\nu\|_\infty < 1, \tag{27}$$
we obtain that $\psi_\lambda(x)$ is a C^1-diffeomorphism in $\mathbb{R}^N$ with
$$\xi_\lambda := \psi_\lambda^{-1} = (I + \lambda\nu)^{-1}, \quad D\xi_\lambda(x) = I + \sum_{k=1}^{\infty}(-\lambda D\nu(x))^k. \tag{28}$$

We assume (A1)–(A3) and make the following additional assumptions on H throughout this section.

(A5) The function H is *geometric*, i.e.,
$$H(x, t, \alpha p, \alpha X + \beta p \otimes p) = \alpha H(x, t, p, X)$$
for all $\alpha > 0$, $\beta \in \mathbb{R}$, $(x, t, p, X) \in \mathbb{R}^N \times [0, T] \times (\mathbb{R}^N \setminus \{0\}) \times \mathcal{S}^N$.

(A6) There exists $L_H > 0$ such that

$$|H(x,t,p,X) - H(x,t,p,Y)| \le L_H |X - Y|$$

for any $(x,t,p) \in \mathbb{R}^N \times [0,T] \times (\mathbb{R}^N \setminus \{0\})$ and $X, Y \in \mathcal{S}^N$.

(A7) For any $R > 0$, there exists $C_H > 0$ such that

$$\begin{aligned} & H\big(\psi_\lambda(y), t, D\xi_\lambda(\psi_\lambda(y))^T p, D\xi_\lambda(\psi_\lambda(y))^T Y D\xi_\lambda(\psi_\lambda(y))\big) - H(x,t,p,X) \\ & \le C_H\big(\frac{|x-y|^4}{\varepsilon^4} + \lambda + \frac{\lambda^2 |x-y|^2}{\varepsilon^4} + \rho \|A^2\|\big) \end{aligned} \tag{29}$$

for any $\lambda \in [0, \lambda_0]$ and for any $\rho, \varepsilon \in (0,1)$, $x, y \in \overline{B}(0,R)$, $t \in [0,T]$, $p = 4\varepsilon^{-4}|x-y|^2(x-y)$, $X, Y \in \mathcal{S}^N$ satisfying (12) and $A \in \mathcal{S}^N$ given by (13).

(A8) There exists at least one viscosity solutions of (21) which satisfies

$$u(x,t) = -1 \quad \text{for all } (x,t) \in (\mathbb{R}^N \setminus B(0,R_T)) \times [0,T] \tag{30}$$

for some $R_T > 0$.

Let us make some comments about these new assumptions: (A5) is needed to use the level set approach to describe front propagation (see [11, 29] for instance). Assumption (A6) is satisfied for a wide class of quasilinear equations under interest in this paper, see Example 2. A consequence of (A5) and (A6) is: For any $R > 0$, there exists $M_R > 0$ such that

$$|H(x,t,p,X)| \le M_R(1 + |X|) \quad \text{on } \overline{B}(0,R) \times [0,T] \times (B(0,R) \setminus \{0\}) \times \mathcal{S}^N, \tag{31}$$

which is a crucial property to obtain Hölder continuity in time for the solutions of (21), see Proposition 3. Assumption (A7) is a natural condition to obtain a preservation of the initial property (I2) during the evolution. This condition is related to (A4); it is worthwhile to notice, as it was done at the end of Example 1, that such a condition gives uniqueness for the solutions of (10). Existence of solutions to (21) is assumed in (A8) because it is not the point in this paper, see [30, 43, 8] for some conditions which guarantee existence. More precisely, we have the following result about solutions of (21) and the proof is given in Appendix:

Proposition 3 (Regularity of Solutions). *There exists a unique viscosity solution $u \in C(\mathbb{R}^N \times [0,T])$ of (21) and we have*

$$|u(x,t) - u(y,t)| \le \|Du_0\|_{L^\infty(\mathbb{R}^N)} e^{Kt} |x - y|, \tag{32}$$

$$|u(x,t) - u(x,s)| \le \tilde{L} |t - s|^{1/2} \tag{33}$$

for all $x, y \in \mathbb{R}^N$, $t, s \in [0,T]$, where $K, \tilde{L}$ are positive constants which depend only on C_H and $C_H, R_T, \|Du_0\|_\infty$ respectively.

Now, we state the main result of this section.

Theorem 4 (Key Estimate)**.** *There exist $\overline{t} > 0$, $0 < \overline{\lambda} \leq \lambda_0$ (λ_0 is given by* (I2) *and satisfies* (27)) *and a non-increasing continuous function $\eta : [0, \overline{t} \wedge T] \to [0, \infty)$ which depend only on C_H, L_H, δ_0, η_0, R_T, $\|Du_0\|_\infty$, $\|\nu\|_\infty$, $\|D\nu\|_\infty$ such that*

$$\eta(t) > 0 \text{ for all } t \in [0, \overline{t} \wedge T),$$

and u satisfies

$$u(x + \lambda\nu(x), t) \geq u(x, t) + \lambda\eta(t) \quad \text{for all } x \in U_t,\ \lambda \in [0, \overline{\lambda}], \tag{34}$$

where $U_t := \{x \in \mathbb{R}^N \mid |u(x, t)| \leq \delta_0/4\}$.

Proof. From (I2) and Lemma 17 in the Appendix, we can extend (22) in $\mathbb{R}^N$,

$$u_0(\psi_\lambda(x)) \geq \Psi(u_0(x) + \lambda\eta_0) \quad \text{for any } x \in \mathbb{R}^N, \lambda \in [0, \overline{\lambda}],$$

where $\overline{\lambda}$ and Ψ are introduced in Lemma 17.

We first prove

$$u(\psi_\lambda(x), t) \geq \Psi(u(x, t) + \lambda\eta_0) - M_2\lambda\sqrt{t} \tag{35}$$

for all $(x, t) \in \mathbb{R}^N \times [0, T]$, $\lambda \in [0, \overline{\lambda}]$, some constant $M_2 > 0$, which is depends only on $C_H, L_H, \delta_0, \eta_0, R_T, \|Du_0\|_\infty$ and $\|\nu\|_\infty$ (Note that M_2 does not depend on $\|D\nu\|_\infty$ contrary to $\overline{\lambda}$ which depends on $\|D\nu\|_\infty$ through λ_0 because of (27)).

Fix $\lambda \in [0, \overline{\lambda}]$. Set $v(x, t) := u(\psi_\lambda(x), t)$ and $w(x, t) := \Psi(u(x, t) + \lambda\eta_0)$ for all $(x, t) \in \mathbb{R}^N \times [0, T]$. Since H is geometric and Ψ is a nondecreasing function, the functions v, w satisfy

$$\begin{cases} v_t + H\big(\psi_\lambda(x), t, D\xi_\lambda(\psi_\lambda(x))^T Dv(x, t), \\ D\xi_\lambda(\psi_\lambda(x))^T D^2 v(x, t) D\xi_\lambda(\psi_\lambda(x)) + D^2\xi_\lambda(\psi_\lambda(x)) Dv(x, t)\big) = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ v(x, 0) = u_0(\psi_\lambda(x)) & \text{in } \mathbb{R}^N, \end{cases}$$

$$\begin{cases} w_t + H(x, t, Dw, D^2 w) = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ w(x, 0) = \Psi(u_0(x) + \lambda\eta_0) & \text{in } \mathbb{R}^N \end{cases}$$

in the viscosity sense (see [29, Theorem 4.2.1] for instance).

Let R_T be the constant in (A8) and recall that $\overline{\lambda}\eta_0 \leq \delta_0/4$ in Lemma 17. For any $(x, t) \in (\mathbb{R}^N \setminus \overline{B}(0, R_T + \overline{\lambda}\|\nu\|_\infty) \times [0, T]$, $u(x, t) + \lambda\eta_0 \leq -1 + \delta_0/4 \leq -(3\delta_0)/4$, which implies that $-1 = \Psi(u(x, t) + \lambda\eta_0) \leq u(\psi_\lambda(x), t)$. Therefore, we only need to show that for any $(x, t) \in \overline{B}(0, R_T + \overline{\lambda}\|\nu\|_\infty) \times [0, T]$ inequality (35) holds. Note that

$$|D^2\xi_\lambda(\psi_\lambda(x))p| \leq C\lambda|p|.$$

By Assumptions (A6), (A7) and Theorem 1 with $\kappa_1 = \lambda$ and $\kappa_2 = \lambda^2$, we get, for any $(x, t) \in \overline{B}(0, R_T + \overline{\lambda}\|\nu\|_\infty) \times [0, T]$,

$$(w - v)(x, t) \leq C(t + \sqrt{t})\lambda \leq C(\sqrt{T} + 1)\sqrt{t}\lambda =: M_2\sqrt{t}\lambda$$

for all $t \in [0, T]$, which implies (35).

Setting $\overline{t} := (\eta_0/M_2)^2$ and $\eta(t) := \eta_0 - M_2\sqrt{t}$ for all $t \in [0, \overline{t} \wedge T]$, we obtain the conclusion. □

The first important consequence is a lower-gradient bound estimate on the front.

Corollary 5 (Estimate on Lower-Bound Gradient)**.** *We have*

$$-|Du(x,t)| \leq -\frac{\eta(t)}{\|\nu\|_\infty} \quad \text{in } \{|u(\cdot,t)| < \frac{\delta_0}{4}\} \times (0, \overline{t} \wedge T),$$

where $\overline{t}$ and η are given in Theorem 4.

Remark 4. Theorem 3.1 in [13] implies that we cannot expect global in time lower gradient estimates for solutions of (59) with general initial data like (I1), (I2), even if we assume some positiveness assumptions on the velocity like in [1, 10, 6, 7].

Before giving the proof of this result, we continue by stating another consequences of Theorem 4. We need to introduce some notations.

For any $t \in [0,T]$, $r \in [-1,1]$, we set

$$\Omega_t^r := \{x \in \mathbb{R}^N \mid u(x,t) > r\}, \quad \Gamma_t^r := \partial\Omega_t^r$$

and define the cone with vertex $z \in \mathbb{R}^N$, axis $e \in \mathbb{S}^{N-1}$ and parameters $(\rho,\theta) \in \mathbb{R}_+ \times \mathbb{R}_+$ by

$$\begin{aligned} C_{e,z}^{\rho,\theta} &:= \bigcup_{a\in[0,\theta]} \overline{B}(z+ae, a\frac{\rho}{\theta}) \\ &= \{z + ae + a\frac{\rho}{\theta}\xi \mid a \in [0,\theta], \xi \in \overline{B}(0,1)\}. \end{aligned}$$

The following result means that the evoluting fronts have the interior cone property.

Corollary 6 (Interior Cone Properties of Fronts)**.** *For any $r \in [-\delta_0/4, \delta_0/4]$ and $t \in [0, \overline{t} \wedge T]$,*

$$C^{\rho(t),\theta(z)}_{\frac{\nu(z)}{|\nu(z)|},z} \subset \overline{\Omega}_t^r \quad \text{for all } z \in \Gamma_t^r,$$

where

$$\rho(t) := \frac{\eta(t)\overline{\lambda}}{\|Du_0\|_\infty e^{Kt}}, \quad \theta(z) := \overline{\lambda}|\nu(z)|.$$

When A is a subset of $\mathbb{R}^k$, we will write, by abuse of notation, $\mathrm{Per}(A) = \mathcal{H}^{k-1}(\partial A)$ for the perimeter of A. Notice that it does not always correspond to the usual definition of perimeter. The two definitions coincide for instance when the boundary is locally the graph of a Lipschitz function, which is often the case in our applications. For further details, see [24, Section 5 and Remark p.183] or [32].

Corollary 7 (Estimate on Perimeter of Fronts)**.** *There exists a constant $M_3 > 0$ which depends only on the constants appearing in Theorem* 4 *such that*

$$\mathrm{Per}\,(\Omega_t^r) \leq M_3$$

for all $r \in [-\delta_0/4, \delta_0/4]$ and $t \in [0, (\overline{t} \wedge T)/2]$.

We turn to the proofs.

Proof of Corollary 5. Take any function $\phi \in C^1(\mathbb{R}^N\times(0,T))$ satisfying $(u-\phi)(x_0,t_0) = 0$ and $(u-\phi)(x,t) \le 0$ for all $(x,t) \in \mathbb{R}^N \times (0,T)$ for some $(x_0,t_0) \in \{|u(\cdot,t)| < \delta_0/4\} \times (0, \overline{t} \wedge T)$, where $\overline{t}$ is given by Theorem 4. By Theorem 4 and mean-value theorem, we have

$$\begin{aligned}\lambda\eta(t_0) &\le u(\psi_\lambda(x_0),t_0) - u(x_0,t_0)\\ &\le \phi(\psi_\lambda(x_0),t_0) - \phi(x_0,t_0)\\ &= \lambda\langle D\phi(x_0,t_0), \nu(x)\rangle + o(\lambda)\\ &\le \lambda|D\phi(x_0,t_0)|\|\nu\|_\infty + o(\lambda\|\nu\|_\infty).\end{aligned}$$

Dividing by $\lambda\|\nu\|_\infty > 0$ in the above and letting $\lambda \in (0,\overline{\lambda}]$ go to 0, we obtain the conclusion. □

Proof of Corollary 6. Fix $r \in [-\delta_0/4, \delta_0/4]$, $t \in [0, \overline{t} \wedge T]$ and $z \in \Gamma_t^r$. By Theorem 4, we have

$$u(z + \lambda\nu(z), t) \ge r + \lambda\eta(t) \quad \text{for all } \lambda \in [0, \overline{\lambda}].$$

Set $r_\lambda(t) := (\lambda\eta(t))/(\|Du_0\|_\infty e^{Kt})$. For any $\xi \in \overline{B}(0,1)$, we have

$$\begin{aligned}u(z + \lambda\nu(z) + r_\lambda(t)\xi, t) \ge\ & u(z+\lambda\nu(z),t) - \|Du_0\|_\infty e^{Kt} r_\lambda(t)\\ &\ge r + \lambda\eta(t) - \|Du_0\|_\infty e^{Kt} r_\lambda(t) \ge r,\end{aligned}$$

which implies that

$$\overline{B}(z + \lambda\nu(z), r_\lambda(t)) \subset \overline{\Omega}_t^r$$

for any $\lambda \in [0,\overline{\lambda}]$. Therefore, we have

$$C^{\rho(t),\theta(z)}_{\frac{\nu(z)}{|\nu(z)|},z} = \bigcup_{\lambda\in[0,\overline{\lambda}]} \overline{B}(z+\lambda\nu(z), \lambda\frac{\rho(t)}{\overline{\lambda}}) = \bigcup_{a\in[0,\theta(z)]} \overline{B}(z + a\frac{\nu(z)}{|\nu(z)|}, a\frac{\rho(t)}{\theta(z)}) \subset \overline{\Omega}_t^r.$$

□

Before doing the proof of Corollary 7, we recall the following lemma.

Lemma 8 ([7, Theorem 5.8])**.** *Let K be a compact subset of $\mathbb{R}^N$ having the interior cone property of parameters ρ and θ. Then there exists a positive constant $\Lambda = \Lambda(N, \rho, \theta/\rho)$ such that for all $R > 0$,*

$$\mathcal{H}^{N-1}(\partial K \cap \overline{B}(0,R)) \le \Lambda\mathcal{L}^N(K \cap \overline{B}(0, R+\rho/4)).$$

Proof of Corollary 7. Set $t^* := (\overline{t} \wedge T)/2$. Let $\rho(t), \theta(z)$ be the functions in Corollary 6 and set $\overline{\rho} := \rho(t^*)$ and $\overline{\theta} := \min_{z\in\partial\Omega_t, t\in[0,t^*]} \theta(z)$. By Theorem 4 and Corollary 6, we see that $\overline{\rho}, \overline{\theta} > 0$ and we have,

$$C^{\overline{\rho},\overline{\lambda}}_{\frac{\nu(z)}{|\nu(z)|},z} \subset C^{\rho(t^*),\theta(z)}_{\frac{\nu(z)}{|\nu(z)|},z} \subset \overline{\Omega}_t^r \quad \text{for all } z \in \Gamma_t^r, r \in [-\frac{\delta_0}{4}, \frac{\delta_0}{4}].$$

Due to Lemma 8, there exists $\Lambda = \Lambda(N, \overline{\rho}, \overline{\theta}/\overline{\lambda}) > 0$ such that

$$\mathcal{H}^{N-1}(\Gamma_t^r) \le \Lambda\mathcal{L}^N(\overline{\Omega}_t^r) \le \Lambda\mathcal{L}^N(B(0,R_T)) =: M_3$$

for all $t \in [0, t^*]$, $r \in [-\delta_0/4, \delta_0/4]$. □

We end this section with an application.

Example 2. We consider the function

$$H(x,t,p,X) = \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \big\{-c^{\alpha,\beta}(x,t,p)|p| - \mathrm{tr}\,\big(\sigma^{\alpha,\beta}(x,t,p)(\sigma^{\alpha,\beta})^T(x,t,p)X\big)\big\}, \quad (36)$$

where the functions $c^{\alpha,\beta}$ and $\sigma^{\alpha,\beta}$ satisfy (20) for all $\alpha\in\mathcal{A}$, $\beta\in\mathcal{B}$, respectively. We add the following assumptions on $c^{\alpha,\beta}, \sigma^{\alpha,\beta}$:

$$\begin{aligned} &c(x,t,\mu p) = c(x,t,p), \quad |c(x,t,p) - c(x,t,q)| \le C_c|p-q|, \\ &\sigma(x,t,\mu p) = \sigma(x,t,p), \quad \sigma^T(x,t,p)p = 0, \quad |\sigma(x,t,p)-\sigma(x,t,q)| \le \frac{C_\sigma|p-q|}{|p|+|q|} \end{aligned} \quad (37)$$

for all $\mu>0$, $(x,t)\in\mathbb{R}^N\times[0,T]$, $p,q\in(\mathbb{R}^N\setminus\{0\})$ and some $C_c, C_\sigma>0$. These assumptions are related to (A5). A typical example is $\sigma^{\alpha,\beta}(x,t,p) = I - p\otimes p/|p|^2$ and then the second-order term is the so-called mean curvature term. We claim that the function H satisfies (A1)–(A8).

It is easy to check that the function H satisfies (A1)–(A5). We check that the function H satisfies (A7). Note that, by (27), (28), we have

$$\begin{aligned} |D\xi_\lambda(\psi_\lambda(x))| &\le \frac{1}{1-\lambda|D\nu(x)|} \le \frac{1}{1-\lambda_0\|D\nu\|_\infty} < +\infty, \\ |I - D\xi_\lambda(\psi_\lambda(x))| &\le \frac{\lambda\|D\nu\|_\infty}{1-\lambda_0\|D\nu\|_\infty}, \end{aligned}$$

for $\lambda_0\in(0,1)$ small enough and any $x\in\mathbb{R}^N$. By abuse of notations, we write c,σ instead of $c^{\alpha,\beta}, \sigma^{\alpha,\beta}$ for any $\alpha\in\mathcal{A}, \beta\in\mathcal{B}$. We compute

$$\begin{aligned} &|c(x,t,p) - c(\psi_\lambda(x), t, D\xi_\lambda(\psi_\lambda(x))^Tp)| \\ \le\,& |c(x,t,p) - c(\psi_\lambda(x),t,p)| + |c(\psi_\lambda(x),t,p) - c(\psi_\lambda(x),t,D\xi_\lambda(\psi_\lambda(x))^Tp)| \\ \le\,& L|x-\psi_\lambda(x)| + C_c|(I - D\xi_\lambda(\psi_\lambda(x))^T)p| \\ \le\,& \tilde{C}\lambda \end{aligned}$$

and

$$\begin{aligned} &|\sigma(x,t,p) - D\xi_\lambda(\psi_\lambda(x))\sigma(\psi_\lambda(x),t,D\xi_\lambda(\psi_\lambda(x))^Tp)| \\ \le\,& |\sigma(x,t,p) - D\xi_\lambda(\psi_\lambda(x))\sigma(x,t,p)| + |D\xi_\lambda(\psi_\lambda(x))(\sigma(x,t,p) - \sigma(\psi_\lambda(x),t,p))| \\ &+ |D\xi_\lambda(\psi_\lambda(x))(\sigma(\psi_\lambda(x),t,p) - \sigma(\psi_\lambda(x),t,D\xi_\lambda(\psi_\lambda(x))^Tp))| \\ \le\,& M|I - D\xi_\lambda(\psi_\lambda(x))| + L|D\xi_\lambda(\psi_\lambda(x))||x-\psi_\lambda(x)| \\ &+ C_\sigma\frac{|D\xi_\lambda(\psi_\lambda(x))||(I - D\xi_\lambda(\psi_\lambda(x))^T)p|}{|p| + |D\xi_\lambda(\psi_\lambda(x))^Tp|} \\ \le\,& \tilde{C}\lambda \end{aligned}$$

for some $\tilde{C} = \tilde{C}(L,M,C_c,C_\sigma)>0$ and any $x\in\mathbb{R}^N, p\in\mathbb{R}^N\setminus\{0\}$ and $\lambda\in[0,\lambda_0]$.

By using the same computations as Example 1, we have

$$\begin{aligned}&H\big(\psi_\lambda(y),t,D\xi_\lambda(\psi_\lambda(y))p,D\xi_\lambda(\psi_\lambda(y))^TYD\xi_\lambda(\psi_\lambda(y))\big)-H(x,t,p,X)\\ \le& C\big(\frac{|x-y|^4}{\varepsilon^4}+\lambda+\frac{\lambda^2|x-y|^2}{\varepsilon^4}+\rho\|A^2\|\big)\end{aligned}$$

for some $C=C(L,M,C_c,C_\sigma)>0$ and any $\rho,\varepsilon\in(0,1)$, $x,y\in\overline{B}(0,R)$, $t\in[0,T]$, $p=\varepsilon^{-4}|x-y|^2(x-y)$ with $x\neq y$, $X,Y\in\mathcal{S}^N$ satisfying (12) and $A\in\mathcal{S}^N$ given by (13), which implies that H satisfies (A7).

We finally check that H satisfies (A8). At first, the constant function -1 is obviously a subsolution of (21). We set $R(t):=M_ct+R_0+\sqrt{2}$, with $M_c\ge\|c\|_\infty$ and define the function $f:\mathbb{R}^N\times[0,T]\to\mathbb{R}$ by

$$f(x,t):=\phi((R(t)-|x|)^2-1),$$

where $\phi(r):=r\vee(-1)$. We prove that f is a viscosity supersolution of (21). It is easily seen that $f(x,0)\ge u_0(x)$ on $\mathbb{R}^N$. Indeed, for all $x\in B(0,R_0)$, we have $f(x,0)\ge(R_0+\sqrt{2}-|x|)^2-1\ge1\ge u_0(x)$ and, for all $x\in\mathbb{R}^N\setminus B(0,R_0)$, $u_0(x)=-1\le f(x,0)$ (see (I1)). We have $f_t(x,t)=2M_c(R(t)-|x|)$, $Df(x,t)=2(|x|-R(t))\frac{x}{|x|}$ and $D^2f(x,t)=2\big(I-\frac{R(t)}{|x|}(I-\frac{x\otimes x}{|x|^2})\big)$ for any $t\in(0,T)$ and $x\in\overline{B}(0,R(t))\setminus\{0\}$. Note that $D^-f(0,t)=\emptyset$ for all $t\in[0,T]$. Set $b^{\alpha,\beta}(x,t,p)=\sigma^{\alpha,\beta}(x,t,p)(\sigma^{\alpha,\beta})^T(x,t,p)$. We calculate that

$$\begin{aligned}&f_t+\inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\{-c^{\alpha,\beta}(x,t,Df)|Df|-\mathrm{tr}\,(b^{\alpha,\beta}(x,t,Df)D^2f)\}\\ =&\inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\Big\{2(M_c-c^{\alpha,\beta}(x,t,Df))(R(t)-|x|)\\ &-2\mathrm{tr}\,\Big(b^{\alpha,\beta}(x,t,Df)\big(I-\frac{R(t)}{|x|}(I-\frac{x\otimes x}{|x|^2})\big)\Big)\Big\}.\end{aligned}$$

Set $e_1:=x/|x|$ and take $e_i\in\mathbb{R}^N$ for $i=2,\dots,N$ so that $\{e_i\}_{i=1,\dots,N}$ is an orthonormal basis. Then we have $(I-\frac{x\otimes x}{|x|^2})e_1=0$ and $(I-\frac{x\otimes x}{|x|^2})e_i=e_i$ for $i=2,\dots,N$. Therefore,

$$\begin{aligned}&\mathrm{tr}\,\Big(b^{\alpha,\beta}(x,t,Df)\big(I-\frac{R(t)}{|x|}(I-\frac{x\otimes x}{|x|^2})\big)\Big)\\ =&\sum_{i=1}^N\big\langle b^{\alpha,\beta}(x,t,Df)\big(I-\frac{R(t)}{|x|}(I-\frac{x\otimes x}{|x|^2})\big)e_i,e_i\big\rangle)\\ =&\sum_{i=1}^N\big\langle b^{\alpha,\beta}(x,t,Df)e_i,e_i\big\rangle-\frac{R(t)}{|x|}\big\langle b^{\alpha,\beta}(x,t,Df)\big(I-\frac{x\otimes x}{|x|^2}\big)e_i,e_i\big\rangle)\\ =&\sum_{i=2}^N\big(1-\frac{R(t)}{|x|}\big)\langle(\sigma^{\alpha,\beta})^T(x,t,Df)e_i,(\sigma^{\alpha,\beta})^T(x,t,Df)e_i\rangle)\\ \le&\,0,\end{aligned}\tag{38}$$

since $|x| < R(t)$ and $(\sigma^{\alpha,\beta})^T(x,t,Df)e_1 = (\sigma^{\alpha,\beta})^T(x,t,-x)x = 0$ by (37). Moreover, $D^-u(0,t) = \emptyset$ and -1 is obviously a supersolution on $\mathbb{R}^N \setminus B(0,R(t))$. Setting $R_T = R(T)$, we see that (A8) is satisfied in view of the comparison theorem for viscosity solutions of (21).

4. Uniqueness of Solutions of Nonlocal Equations

In this section, we consider the initial value problem of the nonlocal and non-monotone geometric equations which is derived from (1), through the *level set approach* (see [19, 25, 29]),

$$\begin{cases} u_t + H[\mathbf{1}_{\{u\geq 0\}}](x,t,Du,D^2u) = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{39}$$

For any function $\chi \in L^\infty(\mathbb{R}^N \times [0,T],[0,1])$, $H[\chi]$ denotes a real-valued function of $(x,t,p,X) \in \mathbb{R}^N \times [0,T] \times (\mathbb{R}^N \setminus \{0\}) \times \mathcal{S}^N$. For almost any $t \in [0,T]$, $(x,p,X) \mapsto H[\chi](x,t,p,X)$ are continuous functions on $\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \times \mathcal{S}^N$ with a possible singularity at $p = 0$. For all $(x,p,X) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \times \mathcal{S}^N$, $t \mapsto H[\chi](x,t,p,X)$ are measurable functions. For any $\chi \in L^\infty(\mathbb{R}^N \times [0,T],[0,1])$, $H[\chi] = H$ satisfies (A2), (A3), (A5).

Furthermore, we make the following assumptions (H1)–(H5-(i)) or (H5-(ii)) and (I1), (I2) on u_0 throughout this section.

(H1) For any $\chi \in L^\infty(\mathbb{R}^N \times [0,T],[0,1])$, equation (39) has a bounded uniformly continuous L^1-viscosity solution $u[\chi]$. Moreover, there exist constants $C, R_T > 0$ independent of $\chi \in L^\infty(\mathbb{R}^N \times [0,T],[0,1])$ such that $|u(x,t)| \leq C$ for all $(x,t) \in \mathbb{R}^N \times [0,T]$ and $u(x,t) = -1$ for all $x \in (\mathbb{R}^N \setminus B(0,R_T)) \times [0,T]$.

(H2) For any $\tau \in [0,T]$ and $\chi \in C([0,\tau];L^1(\mathbb{R}^N))$ such that $\operatorname{supp}\chi(\cdot,t)$ is compact for any $t \in [0,\tau]$, $H[\chi] \in C(\mathbb{R}^N \times [0,\tau] \times (\mathbb{R}^N \setminus \{0\}) \times \mathcal{S}^N)$.

(H3) The functions $H[\chi]$ satisfy (A6) with $H = H[\chi]$ uniformly for any $\chi \in L^\infty(\mathbb{R}^N \times [0,T],[0,1])$.

(H4) The functions $H[\chi]$ satisfy (A7) with $H = H[\chi]$ uniformly for any $\chi \in L^\infty(\mathbb{R}^N \times [0,T],[0,1])$.

(H5-(i)) For any $R > 0$, there exists $C_H > 0$ such that

$$\begin{aligned} &|H[\chi_1](x,t,p,X) - H[\chi_2](y,t,p,Y)| \\ &\leq C_H\Big(\frac{|x-y|^4}{\varepsilon^4} + \kappa_{\chi_1,\chi_2}(x,t) + \frac{\kappa^2_{\chi_1,\chi_2}(x,t)|x-y|^2}{\varepsilon^4} + \rho\|A^2\|\Big) \end{aligned} \tag{40}$$

for any $\chi_1,\chi_2 \in L^\infty(\mathbb{R}^N \times [0,T],[0,1])$ and for any $\rho,\varepsilon \in (0,1)$, $x,y \in \overline{B}(0,R)$, $t \in [0,T]$, $p = \varepsilon^{-4}|x-y|^2(x-y)$ and $X,Y \in \mathcal{S}^N$ satisfying (12) and $A \in \mathcal{S}^N$ given by (13), where

$$\kappa_{\chi_1,\chi_2}(x,t) := \int_{\mathbb{R}^N} |\chi_1(y,t) - \chi_2(y,t)|\,dy.$$

(H5-(ii)) One has inequality (40) by replacing κ_{χ_1,χ_2} by

$$\overline{\kappa}_{\chi_1,\chi_2}(x,t) := \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)|\chi_1(y,s)-\chi_2(y,s)|\,dyds,$$

where $G(x,t)$ is the Green function defined by

$$G(x,t) := \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}.$$

(H6) For any $\chi \in L^\infty(\mathbb{R}^N \times [0,T],[0,1])$, if $\chi_n(x,t) := n\int_t^{t+1/n}\chi(x,s)ds$, then the nonlinearity

$$H_n(x,t,p,X) := H[\chi_n](x,t,p,X)$$

satisfies (A1)-(A4) and $u[\chi_n] \to u[\chi]$ uniformly in $\mathbb{R}^N \times [0,T]$ as $n \to +\infty$.

Assumptions (H1)–(H4) are modifications of (A1)–(A7) in order to be able to deal with the nonlocal equation (39). While (H5-(i)) and (H5-(ii)) are specially designed to encompass dislocation type equations or FitzHugh-Nagumo type systems. Finally (H6) is the assumption which allows to use Theorem 1 through an approximation argument (cf. Remark 2). Further detailed examples are given in Section 5.

We use the following definition of weak solutions introduced in [6] which is inspired by [33, 39, 15, 16].

Definition 1 (Definition of Weak Solutions)**.** *Let $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ be a continuous function. We say that u is a weak solution of* (39) *if there exists $\chi \in L^\infty(\mathbb{R}^N \times [0,T],[0,1])$ such that*

(1) *u is an L^1-viscosity solution of*

$$\begin{cases} u_t + H[\chi](x,t,Du,D^2u) = 0 & \textit{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \textit{in } \mathbb{R}^N, \end{cases} \tag{41}$$

(2) *for almost every $t \in (0,T)$,*

$$\mathbf{1}_{\{u(\cdot,t)>0\}}(x) \le \chi(x,t) \le \mathbf{1}_{\{u(\cdot,t)\ge 0\}}(x) \textit{ for a.e. } x \in \mathbb{R}^N.$$

Moreover, we say that u is a classical solution of (39) *if in addition, for almost all $t \in [0,T]$,*

$$\mathbf{1}_{\{u(\cdot,t)>0\}}(x) = \mathbf{1}_{\{u(\cdot,t)\ge 0\}}(x) \quad \textit{for a.e. } x \in \mathbb{R}^N.$$

Proposition 9 (Weak Solutions are Classical in a Short Time)**.** *If there exists a weak solution $u \in C(\mathbb{R}^N \times [0,T])$ of* (39)*, then u is classical in $\mathbb{R}^N \times (0,\overline{t}\wedge T)$ for some $\overline{t} > 0$ which depends on C_H, L_H, δ_0, η_0, R_T, $\|Du_0\|_\infty$, $\|\nu\|_\infty$, $\|D\nu\|_\infty$.*

Proof. Let $(\chi,u) \in L^\infty(\mathbb{R}^N\times[0,T],[0,1])\times C(\mathbb{R}^N\times[0,T])$ be an L^1-viscosity solution of (41). We prove that

$$\mathbf{1}_{\{u(\cdot,t)>0\}}(x) = \mathbf{1}_{\{u(\cdot,t)\ge 0\}}(x) \text{ for a.e. } (x,t) \in \mathbb{R}^N \times (0,\overline{t}\wedge T) \tag{42}$$

for some $\overline{t} > 0$.

We use (H6) and set $u_n := u[\chi_n]$. We recall that u_n is the viscosity solutions of (21) with $H = H_n$ for all $n \in \mathbb{N}$.

By the comparison theorem for local equations, Proposition 3 we have

$$|u_n(x,t)| \le C \text{ on } \mathbb{R}^N \times [0,T],$$
$$|u_n(x,t)-u_n(y,t)| \le C|x-y|, \quad |u_n(x,t)-u_n(x,s)| \le C|t-s|^{1/2}$$

for all $x, y \in \mathbb{R}^N$, $t, s \in [0,T]$ and some $C > 0$, which is independent of n. In view of Ascoli-Arzelá theorem, the stability (see [15, 16]) and the uniqueness (see [39, 15, 16]) of L^1-viscosity solutions of (41), we have $u_n \to u$ locally uniformly on $\mathbb{R}^N \times [0,T]$ for $u \in C(\mathbb{R}^N \times [0,T])$ which is the L^1-viscosity solution of (41).

Moreover, in view of Corollary 5, we have

$$-|Du_n(x,t)| \le -\frac{\eta(t)}{\|\nu\|_\infty} \quad \text{in } \{|u_n(\cdot,t)| < \frac{\delta_0}{4}\} \times (0, \overline{t} \wedge T)$$

for some $\overline{t} > 0$. By the usual stability result of viscosity solution, we get

$$-|Du(x,t)| \le -\frac{\eta(t)}{\|\nu\|_\infty} \quad \text{in } \{|u(\cdot,t)| < \frac{\delta_0}{4}\} \times (0, \overline{t} \wedge T),$$

which implies that $\mathcal{L}^N(\{u(\cdot,t) = 0\}) = 0$ for a.e. $t \in (0, \overline{t} \wedge T)$ in view of [24, Corollary 1 in p. 84]. Therefore, we get (42). □

Remark 5. By Proposition 9, we have $\mathbf{1}_{\{u(\cdot,t)\ge 0\}} \in C([0,\overline{t} \wedge T]; L^1(\mathbb{R}^N))$ for any weak solution u of (39). In view of (H2), we see that $t \mapsto H[\mathbf{1}_{\{u\ge 0\}}](x,t,p,X)$ is continuous on $[0, \overline{t} \wedge T]$ for any $(x,p,X) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \times \mathcal{S}^N$.

We state our main result.

Theorem 10 (Uniqueness Result of Solutions in a Short Time)**.** *If there exist weak solutions of the initial-value problem* (39)*, they are classical and unique in* $\mathbb{R}^N \times [0, \overline{t}]$*, where* $\overline{t}$ *is given by Theorem* 4*.*

We formulate the main ingredient of the proof of the above theorem as a lemma.

Lemma 11. *Let* $\underline{t} > 0$ *and* η *be a continuous function on* $[0, \underline{t}]$ *such that* $\eta(t) \ge \underline{\eta} > 0$ *for any* $t \in [0, \underline{t}]$ *and* $u : \mathbb{R}^N \times [0, \underline{t}] \to \mathbb{R}$ *be a bounded Lipschitz continuous function with respect to* x *variable which satisfies* (14), (15) *and* (34) *on* $[0, \underline{t}]$*. Then we have*

$$\int_{\mathbb{R}^N} \mathbf{1}_{\{-\delta \le u(\cdot,t) < 0\}}(y)\, dy \le \frac{M_4 \delta}{\underline{\eta}}, \tag{43}$$

$$\int_0^t \int_{\mathbb{R}^N} G(x-y, t-s) \mathbf{1}_{\{-\delta \le u(\cdot,s) < 0\}}(y)\, dyds \le \frac{M_5 \delta}{\underline{\eta}} \tag{44}$$

for any

$$\delta \in (0, \min\{\frac{\delta_0}{4}, \frac{\underline{\eta}\delta_0}{4L\|\nu\|_\infty}, \underline{\eta}\overline{\lambda}\}],$$

where M_4 *is a constant depending on* $N, R, L, \|D\nu\|_\infty$ *and* M_5 *is a constant depending on* $N, R, \lambda_0, \delta_0, L, \|\nu\|_\infty, \|D\nu\|_\infty$*.*

Proof. We first prove the estimate (43). We have

$$\int_{\mathbb{R}^N} \mathbf{1}_{\{-\delta\le u(\cdot,t)<0\}}(y)\,dy \;\;=\;\; \mathcal{L}^N(\{-\delta \le u(\cdot,t)\}) - \mathcal{L}^N(\Omega_t) \tag{45}$$

for $\delta > 0$, since $\Omega_t := \{u(\cdot,t) > 0\} \subset \{-\delta \le u(\cdot,t)\}$.

We claim that

$$\{-\delta \le u(\cdot,t)\} \subset (I + \frac{\delta}{\underline{\eta}}\nu)^{-1}(\Omega_t) \tag{46}$$

for $t \in [0, \underline{t}]$ and δ small enough. We recall that $\psi_\lambda = (I+\lambda\nu)$ is a C^1-diffeomorphism when λ satisfies (27). To prove the claim, let $(x,t) \in \mathbb{R}^N \times [0,\underline{t}]$ such that $u(x,t) \ge -\delta$ and set

$$\lambda := \frac{\delta}{\underline{\eta}}.$$

We distinguish two cases. If $u(x,t) \ge \delta_0/4$, then, by (14),

$$u(x + \lambda\nu(x), t) \ge u(x,t) - \lambda L\|\nu\|_\infty \ge \frac{\delta_0}{4} - \lambda L\|\nu\|_\infty \ge 0$$

for $\lambda \le \delta_0/(4L\|\nu\|_\infty)$. If $-\delta \le u(x,t) \le \delta_0/4$, then, by (34),

$$u(x + \lambda\nu(x), t) \ge u(x,t) + \lambda\eta(t) \ge u(x,t) + \lambda\underline{\eta} \ge -\delta + \lambda\underline{\eta} = 0$$

for $\delta \le \delta_0/4$, $\lambda \le \overline{\lambda}$ and $t \in [0,\underline{t}]$. Finally, (46) holds if δ is such that

$$\delta \le \min\{\frac{\delta_0}{4}, \frac{\underline{\eta}\delta_0}{4L\|\nu\|_\infty}, \underline{\eta}\overline{\lambda}\}.$$

By a change of variable, we have

$$\mathcal{L}^N((I+\lambda\nu)^{-1}(\Omega_t)) = \int_{\Omega_t} \det(D(I+\lambda\nu)^{-1})dx \le (1 + 2N\lambda\|D\nu\|_\infty)\mathcal{L}^N(\Omega_t^i),$$

for small δ and therefore small λ, since

$$\det(D(I+\lambda\nu)^{-1}(x)) = (\det(I + \lambda D\nu(x)))^{-1} = 1 - \lambda\,\mathrm{tr}(D\nu) + o(\lambda).$$

From (45) and (46), it follows

$$\begin{aligned}\int_{\mathbb{R}^N} \mathbf{1}_{\{-\delta\le u(\cdot,t)<0\}}(y)\,dy &\le \mathcal{L}^N((I + \frac{\delta}{\underline{\eta}}\nu)^{-1}(\Omega_t)) - \mathcal{L}^N(\Omega_t)\\ &\le 2N\lambda\|D\nu\|_\infty\mathcal{L}^N(\Omega_t)\\ &\le \frac{2N\|D\nu\|_\infty\mathcal{L}^N(B(0,R))}{\underline{\eta}}\delta =: \frac{M_4\delta}{\underline{\eta}}\end{aligned}$$

by (15).

We next prove the estimate (44). Note that (34) implies the lower gradient estimate

$$-|Du(x,t)| \le -\frac{\underline{\eta}}{\|\nu\|_\infty} \quad \text{in } \{|u(\cdot,t)| < \frac{\delta_0}{4}\} \times (0,\underline{t}).$$

From the increase principle of [10, Lemma 2.3], we get

$$\{-\delta \le u(\cdot,t) < 0\} \subset (\overline{\Omega}_t + \frac{\|\nu\|_\infty \delta}{\underline{\eta}} B) \setminus \overline{\Omega}_t,$$

where $B := B(0,1) \subset \mathbb{R}^N$. Therefore, noting that (34) implies that Ω_t has a interior cone property as we can see in the proof of Corollary 6, by [7, Lemma 4.4] for some M_5 depending on N, R, λ_0, δ_0, $\underline{\eta}$, L, $\|\nu\|_\infty$, $\|D\nu\|_\infty$, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)\mathbf{1}_{\{-\delta \le u(\cdot,s)<0\}}(y)\,dyds \\
\le\ & \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)\mathbf{1}_{(\overline{\Omega}_t + \|\nu\|_\infty \delta B/\underline{\eta})\setminus \overline{\Omega}_t}(y)\,dyds \\
=\ & |\phi(x,t,\frac{\|\nu\|_\infty \delta}{\underline{\eta}}) - \phi(x,t,0)| \le \frac{M_5 \delta}{\underline{\eta}},
\end{aligned}$$

where

$$\phi(x,t,r) := \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)\mathbf{1}_{\overline{\Omega}_s + rB}(y)\,dyds.$$

□

Proof of Theorem 10. Suppose that there exist viscosity solutions u_1 and u_2 of (39). Let $\tau \in (0,T]$ which will be fixed later and set

$$\delta_\tau := \max_{\mathbb{R}^N \times [0,\tau]} |(u_1 - u_2)(x,t)|.$$

In view of Theorem 1 and (H5-(i)) or (H5-(ii)), we have

$$\delta_\tau \le M_1 \kappa_\tau (\tau + \tau^{1/2}) \quad \text{or} \quad \delta_\tau \le M_1 \overline{\kappa}_\tau (\tau + \tau^{1/2}), \tag{47}$$

where

$$\kappa_\tau := \sup_{t \in [0,\tau]} \int_{\mathbb{R}^N} |\mathbf{1}_{\{u_1(\cdot,t)\ge 0\}}(y) - \mathbf{1}_{\{u_2(\cdot,t)\ge 0\}}(y)|\,dy \tag{48}$$

and

$$\overline{\kappa}_\tau := \sup_{x\in\mathbb{R}^N, t \in [0,\tau]} \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s) |\mathbf{1}_{\{u_1(\cdot,s)\ge 0\}}(y) - \mathbf{1}_{\{u_2(\cdot,s)\ge 0\}}(y)|\,dyds. \tag{49}$$

Note that

$$|\mathbf{1}_{\{u_1(\cdot,t)\ge 0\}}(y) - \mathbf{1}_{\{u_2(\cdot,t)\ge 0\}}(y)| \le \mathbf{1}_{\{-\delta_\tau \le u_1(\cdot,t)<0\}}(y) + \mathbf{1}_{\{-\delta_\tau \le u_2(\cdot,t)<0\}}(y).$$

We fix

$$t^* \in (0, \overline{t} \wedge T),$$

where $\overline{t}$ is given by Theorem 4. Take $\tau \le t^*$ small enough in order that $\delta_\tau \le \delta_0/4$, the lower-bound gradient estimate (Corollary 5) holds on $[0,\tau]$ and, for all $t \in [0,\tau]$, $\eta(t) \ge \eta(t^*) =: \overline{\eta} > 0$. Moreover, take $\tau \le t^*$ such that

$$\delta_\tau \in (0, \min\{\frac{\delta_0}{4}, \frac{\overline{\eta}\delta_0 e^{-KT}}{4\|\nu\|_\infty \|Du_0\|}, \overline{\eta}\overline{\lambda}\}],$$

where K is the constant give by Proposition 3. By continuity of u_1, u_2 which achieve the same initial condition u_0, it is always possible to find $\tau > 0$ small enough in order that the above condition holds.

By Lemma 11, we have

$$\kappa_\tau \leq \frac{C_1 \delta_\tau}{\overline{\eta}}$$

for some C_1 depending on $N, R_T, C_H, \|Du_0\|_\infty, \|D\nu\|_\infty$ and

$$\overline{\kappa}_\tau \leq \frac{C_2 \delta_\tau}{\overline{\eta}}$$

for some C_2 depending on $C_H, \lambda_0, \delta_0, \eta_0, \|Du_0\|_\infty \|\nu\|_\infty, \|D\nu\|_\infty$.

Therefore, we get

$$\delta_\tau \leq \frac{C}{\overline{\eta}} \delta_\tau (\tau + \sqrt{\tau})$$

for some constant $C(C_H, \lambda_0, \delta_0, \eta_0, R_T, \|Du_0\|_\infty, N, \|\nu\|_\infty, \|D\nu\|_\infty) > 0$ which is independent of τ. For τ small enough, we have $\delta_\tau = 0$. It follows $u_1 = u_2$ on $\mathbb{R}^N \times [0, \tau]$.

We consider $\overline{\tau} = \sup\{\tau > 0 \mid u_1 = u_2 \text{ on } \mathbb{R}^N \times [0, \tau]\}$. If $\overline{\tau} < t^*$, then we can repeat the above proof from time $\overline{\tau}$ instead of 0. Finally, we have $u_1 = u_2$ on $\mathbb{R}^N \times [0, t^*]$ for all $t^* < \overline{t}$, which gives the conclusion. □

5. Applications

In the companion paper [8], the framework to show existence of weak solutions of (39) is given and as applications, existence results for weak solutions of level set equations appearing in dislocations' theory and in the study of FitzHugh-Nagumo systems are presented (see [8, Sections 3.2, 4.2]). In this section, we give uniqueness results for viscosity solutions of such equations.

5.1. **Dislocation Type Equations.** We consider the level set equation of the evolution of hypersurfaces:

$$V = M(n(x))\big(c_0(\cdot, t) * \mathbf{1}_{\overline{\Omega}_t}(x) + c_1(x, t) - \mathrm{div}_{\Gamma_t}(\xi(n)(x))\big) \quad \text{on } \Gamma_t, \tag{50}$$

where we use the notations in Introduction. Here $M : \mathbb{S}^{N-1} \to \mathbb{R}^+$, $c_0, c_1 : \mathbb{R}^N \times [0, T] \to \mathbb{R}$, $\xi : \mathbb{R}^N \to \mathbb{R}^N$ are given functions which satisfy the following assumption (A):

(i) $M \in C(\mathbb{S}^{N-1})$, $0 \leq M(\hat{p}) \leq \overline{M}$ for all $p \in \mathbb{R}^N \setminus \{0\}$, where $\hat{p} = p/|p|$ and $\hat{p} \mapsto \sqrt{M(\hat{p})}$ is Lipschitz continuous;
(ii) $c_0 \in C([0, T]; L^1(\mathbb{R}^n))$, $c_1 \in C(\mathbb{R}^N \times [0, T])$, $D_x c_0 \in L^\infty([0, T]; L^1(\mathbb{R}^N))$;
(iii) there exist constants $L_c, M_c > 0$ such that, for any $x, y \in \mathbb{R}^N$ and $t \in [0, T]$

$$\|D_x c_0(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq L_c, \quad |c_1(x, t) - c_1(y, t)| \leq L_c |x - y|,$$
$$\|c_0(\cdot, t)\|_{L^1(\mathbb{R}^N)} + |c_0(x, t)| + |c_1(x, t)| \leq M_c;$$

(iv) $\xi(p) = (\xi^1(p), \dots, \xi^N(p)) = D\gamma(p) = (\partial\gamma(p)/\partial p_1, \dots, \partial\gamma(p)/\partial p_N)$ for some positively homogeneous function $\gamma \in C^2(\mathbb{R}^N \setminus \{0\})$ of degree 1, i.e., $\gamma(\alpha p) = \alpha\gamma(p)$ for $\alpha > 0$, $p \in \mathbb{R}^N \setminus \{0\}$, which satisfies

$$D^2\gamma(p) = \begin{pmatrix} \frac{\partial^2\gamma(p)}{\partial p_1 \partial p_1} & \cdots & \frac{\partial^2\gamma(p)}{\partial p_1 \partial p_N} \\ \vdots & & \vdots \\ \frac{\partial^2\gamma(p)}{\partial p_N \partial p_1} & \cdots & \frac{\partial^2\gamma(p)}{\partial p_N \partial p_N} \end{pmatrix} \geq 0 \text{ for all } p \in \mathbb{R}^N \setminus \{0\},$$

$$\sup_{p \in \mathbb{R}^N \setminus \{0\}} |D^2\gamma(\hat{p})| < \infty \text{ and } \hat{p} \mapsto \big(D^2\gamma(\hat{p})^{1/2}\big) \text{ is Lipschitz continuous.}$$

The function ξ is called the *Cahn-Hoffman* vector and the last term in the right hand side of (50) is the anisotropic curvature of Γ_t at x given by $\operatorname{div}_{\Gamma_t}(\xi(n)(x)) = \operatorname{tr}\big((I - n(x) \otimes n(x))D_x(\xi(n))(x)\big)$. We refer the reader to the monograph by Giga [29] and the references therein for more details.

For reader's convenience, we derive the level set equation of (2), see [2, 29]. We have

$$\operatorname{div}_{\Gamma_t}(\xi(n)(x)) = \operatorname{tr}(D_x(\xi(n))(x)) - \operatorname{tr}(n(x) \otimes n(x)D_x(\xi(n))(x)). \tag{51}$$

Since γ is positively homogeneous of degree 1, ξ^i is positively homogeneous of degree 0 for all $i \in \{1, \dots, N\}$, i.e., $\xi^i(\alpha p) = \xi^i(p)$ for all $\alpha > 0$, $p \in \mathbb{R}^N \setminus \{0\}$. Differentiating in α, setting $\alpha = 1$ and noting that $D_p\xi(p) = D^2\gamma(p) \in \mathcal{S}^N$, yields

$$\sum_{j=1}^N \xi^i_{p_j}(p)p_j = \sum_{j=1}^N \xi^j_{p_i}(p)p_j = 0 \quad \text{for all } i \in \{1, \dots, N\}, \tag{52}$$

where $\xi^i_{p_j}(p) = \partial\xi^i(p)/\partial p_j$. Equality (52) yields

$$\operatorname{tr}(n(x) \otimes n(x)D_x(\xi(n))(x)) = \sum_{i,j,k} n^k(x)n^j(x)\xi^j_{p_i}(n(x))n^i_{x_k}(x) = 0,$$

where $n^i_{x_k}(x) = \partial n^i(x)/\partial x_k$, and

$$R_p D^2\gamma(p) = D^2\gamma(p) = D^2\gamma(p)R_p,$$

where $R_p = I - \hat{p} \otimes \hat{p}$. Introducing an auxiliary function $u : \mathbb{R}^N \times [0, T] \to \mathbb{R}$ such that $u(\cdot, t) = 0$ on Γ_t and $u(\cdot, t) > 0$ in Ω_t for all $t \in [0, T]$, we note $n(x) = -Du(x)/|Du(x)|$ and it follows

$$\begin{aligned} \operatorname{div}_{\Gamma_t}(\xi(n)(x)) &= \operatorname{div}_x \xi(-Du(x)) = -\operatorname{tr}\big(D^2\gamma(-Du(x))D^2u(x)\big) \\ &= -\operatorname{tr}\big(R_{Du(x)}D^2\gamma(-Du(x))R_{Du(x)}D^2u(x)\big) \\ &= \frac{-1}{|Du(x)|}\operatorname{tr}\big(R_{Du(x)}D^2\gamma\big(-\frac{Du(x)}{|Du(x)|}\big)R_{Du(x)}D^2u(x)\big). \end{aligned}$$

In the above equalities, we used the homogeneity of degree 0 of ξ and of degree -1 of $D^2\gamma$. Set

$$
\begin{aligned}
c[\chi](x,t,p) &:= M(-\hat{p})\big(c_0(\cdot,t)*\chi(\cdot,t)(x)+c_1(x,t)\big),\\
\sigma(p) &:= \sqrt{M(-\hat{p})}\big(D^2\gamma(-\hat{p})\big)^{1/2}R_p,\\
H_d[\chi](x,t,p,X) &:= -c[\chi](x,t,p)|p| - \operatorname{tr}\big(\sigma(p)\sigma^T(p)X\big),
\end{aligned}
$$

for any $\chi\in L^\infty(\mathbb{R}^N\times[0,T],[0,1])$, $(x,t,p,X)\in\mathbb{R}^N\times[0,T]\times(\mathbb{R}^N\setminus\{0\})\times\mathcal{S}^N$. The level set equation of (50) is the equation

$$
u_t + H_d[\mathbf{1}_{\{u(\cdot,t)\ge 0\}}](x,t,Du,D^2u)=0 \quad \text{in } \mathbb{R}^N\times(0,T),
$$

which is a particular case of (39).

Theorem 12. *Under assumptions* (A), (I1) *and* (I2), *the initial value problem* (39) *with $H=H_d$ has at least a weak solution in $\mathbb{R}^N\times[0,T]$. Moreover, weak solutions are classical and unique in $\mathbb{R}^N\times[0,t^*]$ for some $t^*\in(0,T]$ which depends only on $L_c, M_c, \overline{M}, \delta_0, \eta_0, R_0$, $\|Du_0\|_\infty$ and $\|\nu\|_\infty$.*

We refer to [27, 28] for the short time existence and uniqueness of the solution of a dislocation dynamics equation with a mean curvature term under different assumptions.

Proof. The existence of weak solutions is proved in [8, Theorem 3.3]. By using Theorem 10, we prove a short time uniqueness. It is easy to check that (H2), (H3) are satisfied. Due to the arguments in Example 2 and assumptions (A) (i), (iv), we see that (H4) is satisfied. We prove that H_d satisfies (H1) and (H5-(i)). We first check (H5-(i)). Set $\tilde{c}_i(x,t,p):=c[\chi_i](x,t,p)$ for $(x,t,p)\in\mathbb{R}^N\times[0,T]\times(\mathbb{R}^N\setminus\{0\})$, $\chi_i\in L^\infty(\mathbb{R}^N\times[0,T],[0,1])$ and $i=1,2$. Note that

$$
\begin{aligned}
&|\tilde{c}_1(x,t,p)-\tilde{c}_2(x,t,p)|\\
\le\,& M(-\hat{p})\int_{\mathbb{R}^N}|c_0(x-y)(\chi_1(y,t)-\chi_2(y,t))|\,dy\\
\le\,& \overline{M}M_c\int_{\mathbb{R}^N}|\chi_1(y,t)-\chi_2(y,t)|\,dy.
\end{aligned}
$$

We finally check (H1). Let $(u,\chi)\in C(\mathbb{R}^N\times[0,T])\times L^\infty(\mathbb{R}^N\times[0,T],[0,1])$ be a L^1-viscosity solution of (41). We extend the functions $c[\chi](x,\cdot,p)$ to be equal 0 when $t<0$ and $T\le t$. We set

$$
c^n(x,t,p) := c[\chi](x,\cdot,p)*\zeta_n(t),
$$

for all $(x,t,p)\in\mathbb{R}^N\times[0,T]\times(\mathbb{R}^N\setminus\{0\})$, where $\zeta_n(r):=n\zeta(nr)$ for $r\in\mathbb{R}$ and ζ is a standard mollification kernel. Then we have $c^n\in C(\mathbb{R}^N\times[0,T]\times(\mathbb{R}^N\setminus\{0\}))$,

$$
\Big|\int_0^t c^n(x,s,p)-c[\chi](x,s,p)\,ds\Big|\to 0,
$$

locally uniformly in $(0,T)$ as $n\to\infty$. Moreover, the c^n's are Lipschitz continuous with respect to x variable with the constant L_c and bounded (independently of n). Let u_n be the viscosity solutions of

$$\begin{cases} u_t - c^n(x,t,Du)|Du| - \mathrm{tr}\,(\sigma(Du)\sigma^T(Du)D^2u) = 0 & \text{in } \mathbb{R}^N\times(0,T),\\ u(\cdot,0)=u_0 & \text{in } \mathbb{R}^N\times(0,T),\end{cases}$$

for all $n\in\mathbb{N}$. By Example 2 and Proposition 3, we have

$$\begin{aligned} &|u_n(x,t)|\le 1 \text{ on } \mathbb{R}^N\times[0,T], \quad u_n(x,t)=-1 \text{ on } (\mathbb{R}^N\setminus B(0,R(t)))\times[0,T],\\ &|u_n(x,t)-u_n(y,t)|\le C|x-y|, \quad |u_n(x,t)-u_n(x,s)|\le C|t-s|^{1/2},\end{aligned} \tag{53}$$

for all $x,y\in\mathbb{R}^N$, $t,s\in[0,T]$ and some $C>0$, where $R(t)$ is the function introduced in Example 2. By Ascoli-Arzelá theorem, the uniqueness of L^1-viscosity solutions of (41) with $H[\chi]=H_d[\chi]$ and [4, Theorem 1.1], we have $u_n\to u$ locally uniformly on $\mathbb{R}^N\times[0,T]$ and u still satisfies properties (53).

The above mollification argument can be an alternative way of getting the approximation property we need (cf. (H6)) by a regularization of $H[\chi]$ by $H[\chi](x,\cdot,p,X)*\zeta_n(t)$ instead of $H[\chi*\zeta_n(t)](x,t,p,X)$. □

5.2. **A FitzHugh-Nagumo Type System.** We consider the following system:

$$\begin{cases} u_t = \Big(\alpha(v) + \mathrm{div}\,\big(\dfrac{Du}{|Du|}\big)\Big)|Du| & \text{in } \mathbb{R}^N\times(0,T),\\ v_t - \Delta v = g^+(v)\mathbf{1}_{\{u\ge 0\}} + g^-(v)(1-\mathbf{1}_{\{u\ge 0\}}) & \text{in } \mathbb{R}^N\times(0,T),\\ u(\cdot,0)=u_0,\ v(\cdot,0)=v_0 & \text{in } \mathbb{R}^N,\end{cases} \tag{54}$$

which is obtained as the asymptotic as $\varepsilon\to 0$ of the following Fitzhugh-Nagumo system arising in neural wave propagation or chemical kinetics (see [43, Theorem 4.1]):

$$\begin{cases} u^\varepsilon_t - \Delta u^\varepsilon = \dfrac{1}{\varepsilon^2} f(u^\varepsilon,v^\varepsilon) & \text{in } \mathbb{R}^N\times(0,T),\\ v^\varepsilon_t - \Delta v^\varepsilon = g(u^\varepsilon,v^\varepsilon) & \text{in } \mathbb{R}^N\times(0,T),\end{cases} \tag{55}$$

where

$$\begin{cases} f(u,v) = u(1-u)(u-a)-v & (0<a<1),\\ g(u,v)=u-\gamma v & (\gamma>0).\end{cases}$$

The functions α, g^+ and $g^-:\mathbb{R}\to\mathbb{R}$ appearing in (54) are associated with f and g.

We make the following assumptions (B):

(i) the function v_0 is bounded and of class C^1 with $\|Dv_0\|_\infty<+\infty$;

(ii) the functions g^-, g^+ are Lipschitz continuous on $\mathbb{R}$ with a Lipschitz constant $L_g\ge 0$ and there exist $\underline{g},\overline{g}\in\mathbb{R}$ such that

$$\underline{g}\le g^-(r)\le g^+(r)\le \overline{g} \quad \text{for all } r\in\mathbb{R};$$

(iii) $|\alpha(r)-\alpha(s)|\le L_\alpha|r-s|$, $|\alpha(r)|\le M_\alpha$ for all $r,s\in\mathbb{R}$ and some $L_\alpha, M_\alpha>0$.

For $\chi \in L^\infty(\mathbb{R}^N \times [0,T],[0,1])$, we write v for the solution of

$$\begin{cases} v_t - \Delta v = g^+(v)\chi + g^-(v)(1-\chi) & \text{in } \mathbb{R}^N \times (0,T), \\ v(\cdot,0) = v_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{56}$$

and set $c[\chi](x,t) := \alpha(v(x,t))$. Then Problem (54) reduces to

$$\begin{cases} u_t - \Big(c[\mathbf{1}_{\{u\geq 0\}}](x,t) + \operatorname{div}\big(\frac{Du}{|Du|}\big)\Big)|Du(x,t)| = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{57}$$

which is a particular case of (39).

Theorem 13. *Under assumptions* (B), (I1) *and* (I2)*, the initial value problem* (57) *has at least a weak solution in* $\mathbb{R}^N \times [0,T]$*. Moreover, it is classical and unique in* $\mathbb{R}^N \times [0,t^*]$ *for some* t^* *which depends only on* $L_\alpha, M_\alpha, \delta_0, \eta_0, R_0$, $\|Du_0\|_\infty$ *and* $\|\nu\|_\infty$.

Proof. The existence of weak solutions is proved in [8, Theorem 3.4]. See also [30, 43]. It is easy to check that (H2) and (H3) are satisfied. Due to similar arguments as those in the proof of Theorem 12, we see that (H1) and (H4) are satisfied. We prove that (H5-(ii)) is satisfied. For $\chi_1, \chi_2 \in L^\infty(\mathbb{R}^N \times [0,T],[0,1])$, the solutions of (56) are given by

$$\begin{aligned} v_i(x,t) = &\int_{\mathbb{R}^N} G(x-y,t)v_0(y)\,dy + \\ &\int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)(g^+(v_i)\chi_i(y,s) + g^-(v_i)(1-\chi_i(y,s)))\,dyds \end{aligned}$$

for $i = 1,2$. By the proof of [7, Theorem 4.1], we have

$$|c[\chi_1](x,t) - c[\chi_2](x,t)| \leq L_\alpha(\overline{g} - \underline{g})e^{3L_gT} \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)|\chi_1(y,s) - \chi_2(y,s)|\,dyds$$

for all $(x,t) \in \mathbb{R}^N \times [0,T]$. This completes the proof. □

5.3. **Nonlocal equations with volume-dependent terms.** We consider the following evolution of hypersurfaces:

$$V = \beta(\mathcal{L}^N(\overline{\Omega}_t)) - \operatorname{div}_{\Gamma_t}(n(x)) \quad \text{on } \Gamma_t, \tag{58}$$

where the function $\beta : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. A typical example is $\beta(r) = a + br$ for some $a, b \in \mathbb{R}$ which has been studied by Chen, Hilhorst and Logak in [20] (see [17, 18] also). The authors prove that the limiting behaviour of the following reaction-diffusion equation

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2} f(u, \varepsilon \displaystyle\int_\Omega u) & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Omega \times (0,\infty), \\ u(\cdot,0) = g^\varepsilon & \text{in } \Omega \end{cases}$$

is characterized by the motion of hypersurface (58) with $\beta(r) = a + br$. The level set equation of (58) is the nonlocal equation:

$$\begin{cases} u_t - \Big(\beta\big(\mathcal{L}^N(\{u(\cdot,t) \geq 0\})\big) + \gamma \operatorname{div}\big(\dfrac{Du}{|Du|}\big)\Big)|Du| = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{59}$$

for some positive constant γ.

It is worthwhile to mention that we cannot expect the global existence of weak solutions without any restriction of the growth of β, because the front can blow up at a finite time. A growth condition to ensure a global existence result is given below in (C-(ii)), see also [8, Section 4.2].

We can easily check that the equation (59) satisfies assumptions (H1)-(H4) and (H5-(i)). So, the following result holds.

Theorem 14. *Under assumptions* (I1) *and* (I2)*, for any $\gamma, R > 0$, there exists a constant $t_R \in (0,T]$ and at least a weak solution $(u,\chi) \in C(\mathbb{R}^N \times [0,T]) \times L^\infty(\mathbb{R}^N \times [0,T],[0,1])$ of the initial value problem for* (59) *such that $\{x \in \mathbb{R}^N \mid u(x,t) \geq 0\} \subset B(0,R)$ for any $t \in [0,t_R]$. Moreover, the weak solution is classical and unique in $\mathbb{R}^N \times [0,t^*]$ for some $t^* \in (0,t_R]$ which depends only on the Lipschitz constant of β, R, N, δ_0, η_0, R_0, $\|Du_0\|_\infty$ and $\|\nu\|_\infty$.*

Now, adding the assumption (C):

(i) the assumption (I2) holds with $\nu(x) = -x$ in U_0,
(ii) there exist $L_1, L_2 > 0$ such that

$$0 \leq \beta(r) \leq L_1 + L_2 r^{1/N} \quad \text{for all } r \in [0,\infty),$$

we can show a global existence and uniqueness result of solutions of (59) for small γ. More precisely, we get

Theorem 15. *Under assumptions* (I1), (I2) *and* (C)*, there exists a positive constant $\overline{\gamma} = \overline{\gamma}(N,T,\eta_0/\|Du_0\|_\infty)$ such that, for any $0 \leq \gamma \leq \overline{\gamma}$, there exists a unique viscosity solution of* (59) *in $\mathbb{R}^N \times [0,T]$.*

Lemma 16. *Let u be the viscosity solution of*

$$\begin{cases} u_t = c(t)|Du| + \gamma \operatorname{tr}\big((I - \dfrac{Du \otimes Du}{|Du|^2})D^2u\big) & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

where γ is a positive constant and $c \in C([0,T])$ is a nonnegative given function. Then we have, for any $(x,t) \in \{|u(\cdot,t)| \leq \delta_0/4\} \times [0,T]$,

$$u((1-\lambda)x,t) \geq u(x,t) + (\eta_0 - C\|Du_0\|_\infty\sqrt{\gamma t})\lambda \quad \text{for all } \lambda \in [0,\overline{\lambda}],$$

where C is a positive constant which depends only on N and H and $\overline{\lambda}$ is the constant given by Lemma 17 *(We may assume that $\overline{\lambda} \leq 1/2$).*

The proof of Lemma 16 is very similar to that of Theorem 4, but since we would like to explain how to use the positiveness of $c(t)$ and note the dependence of C in Lemma 16, we give it here.

Proof. Let Ψ be the function given by Lemma 17 and fix $\lambda \in [0, \overline{\lambda}]$. Setting $v(x,t) := u((1-\lambda)x, t)$ and $w(x,t) := \Psi(u(x,t) + \eta_0\lambda)$ for all $(x,t) \in \mathbb{R}^N \times [0,T]$, we consider

$$\sup_{x,y\in\mathbb{R}^N, t\in[0,T]} \{w(x,t) - v(y,t) - \frac{|x-y|^4}{\varepsilon^4} - \tilde{K}t\}.$$

for $\varepsilon, \tilde{K} > 0$. By similar arguments as those used in Theorem 1, there exist $(\overline{x}, \overline{y}, \overline{t}) \in \overline{B}(0, R_T+1)^2 \times [0,T]$ for some $R_T > 0$ and small enough $\varepsilon > 0$ such that the supremum attains at $(\overline{x}, \overline{y}, \overline{t})$ and $(a, p, X) \in \overline{J}^{2,+} v(\overline{x}, \overline{t})$ and $(b, p, Y) \in \overline{J}^{2,-} w(\overline{y}, \overline{t})$ satisfy (17). Note that the Lipschitz constant of u is $\|Du_0\|_\infty$ in this case. We have

$$|\overline{x} - \overline{y}| \le \|Du_0\|_\infty^{1/3} \varepsilon^{4/3}, \quad |p| \le 4\|Du_0\|_\infty.$$

We only consider the case where $\overline{x} \neq \overline{y}$. The definition of viscosity solutions immediately implies that we have

$$\begin{aligned} a &\le c(\overline{t})|p| + \gamma \operatorname{tr}\big((I - \frac{p\otimes p}{|p|^2})X\big), \\ b &\ge \frac{c(\overline{t})}{1-\lambda}|p| + \frac{\gamma}{(1-\lambda)^2}\operatorname{tr}\big((I - \frac{p\otimes p}{|p|^2})Y\big). \end{aligned}$$

It follows

$$a - b = \tilde{K} \le c(\overline{t})(1 - \frac{1}{1-\lambda})|p| + \gamma \operatorname{tr}\big((I - \frac{p\otimes p}{|p|^2})(X - \frac{Y}{(1-\lambda)^2})\big).$$

We note that $I - p\otimes p/|p|^2$ is a positive definite bounded matrix and

$$X - \frac{Y}{(1-\lambda)^2} \le \left(\frac{-\lambda}{1-\lambda}\right)^2 \frac{|\overline{x}-\overline{y}|^2}{\varepsilon^4} I + \rho \sum_{i=1}^N \big\langle A^2 \begin{pmatrix} e_i \\ e_i \end{pmatrix}, \begin{pmatrix} e_i \\ e_i \end{pmatrix} \big\rangle$$

by (12). In view of the positiveness of $c(t)$, we get, for some constant $C = C(H,N) > 0$ which may change line to line,

$$\begin{aligned} \tilde{K} &\le \frac{C\gamma\lambda^2}{(1-\overline{\lambda})^2} \frac{|\overline{x}-\overline{y}|^2}{\varepsilon^4} + C\rho\|A^2\| \\ &\le \frac{C\|Du_0\|_\infty^{2/3}\gamma\lambda^2}{\varepsilon^{4/3}} + C\rho\|A^2\|. \end{aligned}$$

Sending $\rho \to 0$ and taking $\tilde{K} = (C+1)\|Du_0\|_\infty^{2/3}\gamma\lambda^2\varepsilon^{-4/3}$, we necessarily have $\overline{t} = 0$.

Thus we have for any $(x,t) \in \mathbb{R}^N \times [0,T]$

$$\begin{aligned} &w(x,t) - v(x,t) \le w(\overline{x}, 0) - v(\overline{y}, 0) + \tilde{K}t \le v(\overline{x}, 0) - v(\overline{y}, 0) + \tilde{K}t \\ &\le e^{4KT/3}\|Du_0\|_\infty^{4/3}\varepsilon^{4/3} + (C+1)\|Du_0\|_\infty^{2/3}\gamma\lambda^2 t\varepsilon^{-4/3}. \end{aligned}$$

Setting

$$\varepsilon^{4/3} = \Big(\frac{(C+1)\gamma\lambda^2 t}{e^{4KT/3}\|Du_0\|_\infty^{2/3}}\Big)^{1/2},$$

we get

$$w(x,t) - v(x,t) \leq \tilde{C}\|Du_0\|\sqrt{\gamma t}\lambda,$$

where $\tilde{C}$ depends only on N, H. This implies a conclusion. $\square$

Proof of Theorem 15. In [8, Section 4], the global existence result for weak solutions of (59) is given. Due to Lemma 16, we see that if

$$\gamma \leq \frac{1}{T}\Big(\frac{\eta_0}{2C\|Du_0\|_\infty}\Big)^2,$$

then

$$\eta_0 - C\|Du_0\|_\infty\sqrt{\gamma t} \geq \frac{\eta_0}{2} \quad \text{for all } t \in [0,T].$$

Therefore, we get

$$u((1-\lambda)x,t) \geq u(x,t) + \frac{\eta_0}{2}\lambda \quad \text{for all } (x,t) \in \{|u(\cdot,t)| \leq \delta_0/4\} \times [0,T],\ \lambda \in [0,\overline{\lambda}]. \tag{60}$$

A careful review of the proof of Theorems 10 gives the conclusion. $\square$

Remark 6. Inequality (60) implies that the r-level set of u, for r close to 0, are star-shaped domains with respect to a ball with center 0, see Lemma 2. In particular, they are locally Lipschitz continuous graphs. Then we can get perimeter estimates without using Lemma 8. Indeed, noting that, from the lower gradient estimate (Corollary 5) and the increase principle [10, Lemma 2.3], for small $\varepsilon > 0$, $\{-\varepsilon \leq u(\cdot,t) \leq 0\} \subset \frac{\overline{\eta}}{\overline{\eta}-\varepsilon}\overline{\Omega}_t \setminus \Omega_t$, we have, for any $t \in [0, \overline{t} \wedge T]$,

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{-\varepsilon}^{0} \mathcal{H}^{N-1}(\{u(\cdot,t)=r\})\,dr &= \frac{1}{\varepsilon}\int_{\{-\varepsilon<u(\cdot,t)<0\}} |Du(x,t)|\,dx \\
&\leq \frac{\|Du_0\|_\infty e^{Kt}}{\varepsilon}\mathcal{L}^N(\{-\varepsilon < u(\cdot,t) < 0\}) \\
&\leq \frac{\|Du_0\|_\infty e^{Kt}}{\varepsilon}\mathcal{L}^N(\frac{\overline{\eta}}{\overline{\eta}-\varepsilon}\Omega_t \setminus \Omega_t) \\
&= \frac{\|Du_0\|_\infty e^{Kt}}{\varepsilon}\big((\frac{\overline{\eta}}{\overline{\eta}-\varepsilon})^N - 1\big)\mathcal{L}^N(\Omega_t)
\end{aligned}$$

in view of the co-area formula and Proposition 3. Since the $\{u(\cdot,t) = r\}$'s are locally Lipschitz continuous, the $(N-1)$-Hausdorff measure and the perimeter in Geometric Measure Theory coincide (see [24]). Since this latter perimeter is lower-semicontinuous with respect to the Hausdorff convergence, sending $\varepsilon \to 0$, we get

$$\mathcal{H}^{N-1}(\{u(\cdot,t)=0\}) \leq \frac{N\|Du_0\|_\infty e^{Kt}}{\overline{\eta}}\mathcal{L}^N(\Omega_t)$$

for all $t \in [0,T]$.

6. Appendix

We give the proof of Lemma 2, Proposition 3 and Lemma 17.

Proof of Lemma 2. Let Ω_0 be defined by (24) and consider the function

$$u_0(x) = \begin{cases} \mathrm{dist}(x,\Gamma_0)\wedge 1 & x\in\overline{\Omega}_0, \\ (-\mathrm{dist}(x,\Gamma_0))\vee(-1) & x\in\mathbb{R}^N\setminus\Omega_0. \end{cases}$$

It is not difficult to see that (25), (I1) and (I2) hold with $\nu(x)=-x$. Conversely, suppose that (I1) and (I2) with $\nu(x)=-x$ hold for some u_0. We claim that $\Omega_0=\{u_0>0\}$ is star-shaped with respect to the ball $B(0,r_0)$ with

$$r_0<\frac{\eta_0}{\|Du_0\|_\infty}.$$

Let $y\in\overline{B}(0,r_0)$ and $x\in\partial\Omega_0$ and define $g(\lambda)=u_0((1-\lambda)x+\lambda y)$. It suffices to show that $g>0$ on $(0,1]$. From (I1), (I2), we have

$$g(\lambda)=u_0((1-\lambda)x+\lambda y)\geq u_0((1-\lambda)x)-\lambda\|Du_0\|_\infty r_0\geq\eta_0\lambda-\lambda\|Du_0\|_\infty r_0>0$$

for $\lambda\in(0,\lambda_0]$. Let

$$\lambda^*=\max\{\lambda\in[\lambda_0,1]:g>0\text{ on }[0,\lambda]\}.$$

If $\lambda^*=1$, then the proof is complete. Otherwise, let $\varepsilon>0$ small enough such that $\varepsilon/(1-\lambda^*+\varepsilon)<\lambda_0$. From (I1), (I2), we have

$$\begin{aligned} 0=g(\lambda^*) &= u_0((1-\lambda^*)x+\lambda^*y) \\ &= u_0((1-\frac{\varepsilon}{1-\lambda^*+\varepsilon})((1-\lambda^*+\varepsilon)x+(\lambda^*-\varepsilon)y)+\frac{\varepsilon}{1-\lambda^*+\varepsilon}y) \\ &\geq g(\lambda^*-\varepsilon)+(\eta_0-\|Du_0\|_\infty r_0)\frac{\varepsilon}{1-\lambda^*+\varepsilon}>0, \end{aligned}$$

which is a contradiction. It completes the proof of the claim. The proof of the fact that a star-shaped with respect to a ball domain has a locally Lipschitz continuous boundary may be found in [32, Prop. 2.4.4 and Theorem 2.4.7] or [38, Lemma p.20].

We turn to the proof of (ii). We need to recall some notations and definitions and we refer the reader to [21] for further details. The Clarke generalized directional derivative at x in the direction h of $f:\mathbb{R}^N\to\mathbb{R}$ is

$$f^\circ(x,h)=\limsup_{y\to x,\lambda\downarrow 0}\frac{f(y+\lambda h)-f(y)}{\lambda}.$$

The Clarke generalized derivative at x is the closed convex set

$$\partial f(x)=\{p\in\mathbb{R}^N\ :\ \text{for all } h\in\mathbb{R}^N,\ \limsup_{y\to x,\lambda\downarrow 0}\frac{f(y+\lambda h)-f(y)-\langle p,h\rangle}{\lambda}\geq 0\},$$

which is nonempty when f is locally Lipschitz continuous at x. The Clarke tangent cone to Ω at $x\in\Gamma$ is the convex cone

$$T_\Omega(x)=\{h\in\mathbb{R}^N: d_\Omega^\circ(x,h)=0\}$$

and the Clarke normal cone is the polar of the latter, i.e.,

$$N_\Omega(x) = \{\xi \in \mathbb{R}^N : \langle \xi, h\rangle \leq 0 \text{ for all } h \in T_\Omega(x)\}.$$

The proof is divided in several steps.
1. We claim that, for any $x \in \Gamma$ and $\xi \in N_\Omega(x) \cap \mathbb{S}^{N-1}$, there exists $\nu_x \in \mathbb{S}^{N-1}$ such that

$$1 \geq \langle \xi, \nu_x\rangle \geq \frac{1}{\sqrt{K^2+1}}. \tag{61}$$

We fix $x \in \Gamma$, $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and consider neighborhoods $V' = B(x', r)$ of x' and $V_N = B(x_N, r)$ of x_N such that there exists a K-Lipschitz continuous function $f : V' \to \mathbb{R}$ with $\mathrm{graph}(f) = \Gamma \cap (V' \times V_N)$ and $\mathrm{epi}\, f \cap (V' \times V_N) = \Omega \cap (V' \times V_N)$. Let $\xi = (\xi', \xi_N) \in N_\Omega(x) \cap \mathbb{S}^{N-1} = N_{\mathrm{epi}\, f}(x', f(x')) \cap \mathbb{S}^{N-1}$. By definition, for all $v = (v', v_N) \in T_{\mathrm{epi}\, f}(x', f(x'))$,

$$\langle (v', v_N), (\xi', \xi_N)\rangle = \langle v', v_N\rangle + v_N \xi_N \leq 0. \tag{62}$$

By [21, Theorem 2.5.7], since f is Lipschitz continuous, $T_{\mathrm{epi}\, f}(x', f(x')) = \mathrm{epi}\, f^\circ(x; \cdot)$. It follows that, for any $r \geq v_N$, (v', r) still belongs to $T_{\mathrm{epi}\, f}(x', f(x'))$. From (62), we get

$$\langle v', v_N\rangle + r\xi_N \leq 0 \quad \text{for all } r \geq v_N. \tag{63}$$

A first consequence is that $\xi_N \leq 0$ necessarily. Moreover

$$\xi_N < 0.$$

Indeed, if $\xi_N = 0$ then (63) holds for all $r \in \mathbb{R}$ and therefore $(v', r) \in T_{\mathrm{epi}\, f}(x', f(x')) = \mathrm{epi}\, f^\circ(x; \cdot)$ for all $r \in \mathbb{R}$. Therefore, using that f is K-Lipschitz continuous, we get

$$r \geq f^\circ(x; v') \geq -K|v'|$$

which leads to a contradiction for $r \to -\infty$. Using again [21, Theorem 2.5.7], we have

$$(\xi', \xi_N) = -\xi_N\left(-\frac{\xi'}{\xi_N}, -1\right) \in N_{\mathrm{epi}\, f}(x', f(x')) \quad \Rightarrow \quad -\frac{\xi'}{\xi_N} \in \partial f(x).$$

Since f is K-Lipschitz continuous, it follows

$$\frac{|\xi'|}{|\xi_N|} \leq K$$

Using that $\xi \in \mathbb{S}^{N-1}$, we obtain

$$|\xi_N| \geq \frac{1}{\sqrt{1+K^2}}.$$

Taking $\nu_x = (0, -1)$ we obtain easily (61).

We denote by d_Ω the distance to Ω and by d^s_Γ the signed distance to Γ which is negative in Ω. For any set $A \subset \mathbb{R}^N$, $\overline{\mathrm{co}}\, A$ is the closure of the convex hull of A.

2. We claim $\partial d^s_\Gamma(x) \subset \overline{\mathrm{co}}[N_\Omega(x) \cap \mathbb{S}^{N-1}]$ for all $x \in \Gamma$. (Notice it means that the generalized derivative of the signed distance does not contain 0.) Let $x \in \Gamma$ and x_i a sequence of points which converges to Γ such that d_Γ^s is differentiable at x_i. Assume that $p_i := \nabla d_\Gamma^s(x_i)$ converges to $\overline{p}$. Suppose first that, up to extract a subsequence, $x_i \notin \overline{\Omega}$. Then, since $d_\Gamma^s = d_\Omega$ in $\mathbb{R}^N \setminus \overline{\Omega}$, and d_Ω is differentiable at x_i, it means that x_i has a unique closest point $\overline{x}_i \in \Gamma$ and

$$p_i = \nabla d_\Omega(x_i) = \frac{x_i - \overline{x}_i}{|x_i - \overline{x}_i|}, \quad |p_i| = 1.$$

But $N_\Omega(x)$ is the convex hull of the cone generated by such limits ([21, Exercise 8.5, p.96]). Thus $\overline{p} \in N_\Omega(x) \cap \mathbb{S}^{N-1}$. From [21, Theorem 2.8.1], $\partial d_\Gamma^s(x)$ is the convex hull of such $\overline{p}$ which completes the proof of the claim.

3. There exists an open bounded neighborhood $\mathcal{O}$ of Γ and $\bar\eta > 0$ such that, for all $y \in \mathcal{O}$,

$$\mathcal{V}(y) = \{\nu \in \overline{B(0,1)} : 1 \geq \langle p, \nu\rangle \geq \bar\eta \ for\ all\ p \in \partial d_\Gamma^s(y)\}$$

is a nonempty compact convex subset of $\mathbb{R}^N$. The subsets $\mathcal{V}(y)$ are clearly convex, closed and bounded for any $1 > \bar\eta > 0$ and open subset $\mathcal{O}$. It remains to prove that there are nonempty for some $\bar\eta$. It is true for $\mathcal{V}(x)$ with $\bar\eta = \sqrt{1+K^2}^{-1}$ by Claims 1 and 2. To extend this property in a neighborhood $B(x,r)$ of x, we notice the following facts: since $\partial d_\Gamma^s(\cdot)$ is upper-semicontinuous ([21, Proposition 2.1.5]) and $0 \notin \partial d_\Gamma^s(x)$, there exists $r > 0$ such that $0 \notin \partial d_\Gamma^s(y)$ for all $y \in B(x,r)$. By Clarke's implicit function theorem [21, Proposition 3.3.6], the $d_\Gamma^s(y)$-level sets of d_Γ^s are Lipschitz continuous in $B(x,r)$. The Lipschitz constant is controlled by the distance from 0 to $\partial d_\Gamma^s(y)$. Up to take r small, it depends only on K. Then, we can repeat the previous arguments and obtain the result in $B(x,r)$ (up to take $0 < \bar\eta$ smaller than $1/\sqrt{1+K^2}$). We then find $\mathcal{O}$ by compactness of Γ.

4. The multi-valued map $\mathcal{V} : \mathcal{O} \rightrightarrows \mathbb{R}^N$ is lower-semicontinuous. Let $y \in \mathcal{O}$. Let $\nu \in \mathcal{V}(y)$ and $\epsilon > 0$. Since $\partial d_\Gamma^s(\cdot)$ is upper-semicontinuous, by definition, there exists $\delta > 0$ such that, if $|y - y'| < \delta$ then $\partial d_\Gamma^s(y') \subset \partial d_\Gamma^s(y) + \epsilon B$ (where $B = \overline{B(0,1)}$). Any $p' \in \partial d_\Gamma^s(y')$ can be written $p' = p + \epsilon w$ where $p \in \partial d_\Gamma^s(y)$ and $|w| \leq 1$. Using that $\nu \in \mathcal{V}(y)$, it follows

$$1 + \epsilon \geq \langle \nu, p'\rangle = \langle \nu, p\rangle + \langle \nu, w\rangle \geq \bar\eta - \epsilon$$

This proves that $\mathcal{V}(y) \subset \mathcal{V}(y') + \epsilon B$ which is the definition of the lower-semicontinuity of a multi-valued function.

5. Michael's continuous selection theorem and end of the proof. From steps 3 and 4, we can apply Michael's continuous selection theorem ([3]): there exists a continuous map $\Phi : \mathcal{O} \to \mathbb{R}^N$ such that $\Phi(y) \in \mathcal{V}(y)$, i.e., for all $y \in \mathcal{O}$ and $p \in \partial d_\Gamma^s(y)$, we have

$$d_\Gamma^s(y + \lambda\Phi(y)) \geq d_\Gamma^s(y) + \langle p, \lambda\Phi(y)\rangle + o_y(\lambda) \geq d_\Gamma^s(y) + \eta\lambda + o_y(\lambda). \tag{64}$$

According to Remark 3, up to decrease η, we may choose Φ which is smooth, bounded and Lipschitz continuous. Using the Lipschitz continuity of d_Γ^s and Φ, we may earn some uniformity in (64) up to reduce η. More precisely, for every $x \in \mathcal{O}$, there exists r_x and λ_x such that

$$d_\Gamma^s(y + \lambda\Phi(y)) \geq d_\Gamma^s(y) + \frac{\bar\eta}{2}\lambda,$$

for all $y \in B(x, r_x) \subset \mathcal{O}$ and $\lambda \in [0, \lambda_x]$. We conclude by compactness of Γ (Note that we can modify the signed distance function far from Γ in order to have a bounded function). $\square$

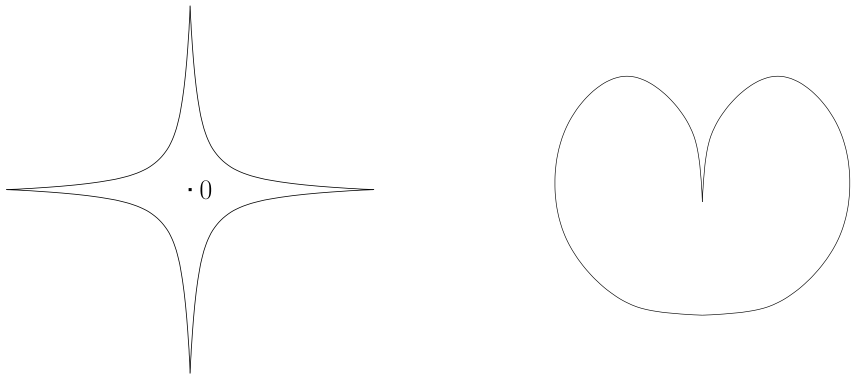

FIGURE 1. *A set which is star-shaped with respect to 0 but does not satisfy (I2) and a set whose boundary is not locally Lipschitz continuous and which satisfies (I2).*

Remark 7. Star-shaped property is not sufficient to ensure (I2), see the counterexample of Figure 1. There exist some sets which satisfy (I2) but they do not have a locally Lipschitz boundary, see Figure 1.

Proof of Proposition 3. Existence of a solution to (21) is given by Assumption (A8). We prove the uniqueness and the Lipschitz continuity regularity in x of the solutions. Let $u \in C(\mathbb{R}^N \times [0, T])$ be a solution of (21) which satisfy (30). Let $K, \varepsilon, \eta > 0$ and set

$$M := \sup_{x,y \in \mathbb{R}^N, t \in [0,T]} \{u(x,t) - u(y,t) - e^{Kt}\frac{|x-y|^4}{\varepsilon^4} - \eta t\}.$$

Let M be attained at $(\hat{x}, \hat{y}, \hat{t}) \in \mathbb{R}^{2N} \times [0, T]$. By (A8), we may assume that $(\overline{x}, \overline{y}, \overline{t}) \in \overline{B}(0, R)^2 \times [0, T]$.

We first consider the case where $\overline{t} \in (0, T]$. In view of Ishii's lemma, for any $\rho > 0$, there exist $(a, p, X) \in \overline{J}^{2,+} u(\overline{x}, \overline{t})$ and $(b, p, Y) \in \overline{J}^{2,-} u(\overline{y}, \overline{t})$ such that

$$a - b = \frac{Ke^{K\overline{t}}|\overline{x} - \overline{y}|^4}{\varepsilon^4} + \eta, \quad p = \frac{4e^{K\overline{t}}|\overline{x} - \overline{y}|^2}{\varepsilon^4}(\overline{x} - \overline{y}),$$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \rho A^2, \tag{65}$$

where

$$A := \frac{4e^{K\overline{t}}}{\varepsilon^4}|\overline{x}-\overline{y}|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \frac{8e^{K\overline{t}}}{\varepsilon^4} \begin{pmatrix} (\overline{x} - \overline{y}) \otimes (\overline{x} - \overline{y}) & -(\overline{x} - \overline{y}) \otimes (\overline{x} - \overline{y}) \\ -(\overline{x} - \overline{y}) \otimes (\overline{x} - \overline{y}) & (\overline{x} - \overline{y}) \otimes (\overline{x} - \overline{y}) \end{pmatrix}.$$

The definition of viscosity solutions immediately implies the following inequalities:

$$a + H_*(\overline{x}, \overline{t}, p, X) \leq 0, \quad b + H^*(\overline{y}, \overline{t}, p, Y) \geq 0.$$

We have

$$\frac{Ke^{K\overline{t}}|\overline{x} - \overline{y}|^4}{\varepsilon^4} + \eta + H_*(\overline{x}, \overline{t}, p, X) - H^*(\overline{y}, \overline{t}, p, Y) \leq 0. \tag{66}$$

We shall distinguish two cases: (i) for any $\varepsilon \in (0,1)$, $p \neq 0$; (ii) there exist $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that $\varepsilon_j \to 0$ as $j \to \infty$, $p = 0$ for any $j \in \mathbb{N}$.

We first consider case (i). By (A7) with $\lambda = 0$, we have

$$\begin{aligned}\frac{Ke^{K\overline{t}}|\overline{x} - \overline{y}|^4}{\varepsilon^4} + \eta &\leq H(\overline{y}, \overline{t}, p, Y) - H(\overline{x}, \overline{t}, p, X)\\ &= H(\overline{y}, \overline{t}, \frac{4e^{K\overline{t}}|x-y|^2}{\varepsilon^4}(x-y), Y) - H(\overline{x}, \overline{t}, \frac{4e^{K\overline{t}}|x-y|^2}{\varepsilon^4}(x-y), X)\\ &\leq C_H\Big(\frac{e^{K\overline{t}}|\overline{x} - \overline{y}|^4}{\varepsilon^4} + \rho\|A\|^2\Big).\end{aligned}$$

In case (ii), we have $\overline{x} = \overline{y}$. Therefore we have $A = 0$, $X \leq 0$, $Y \geq 0$ and $\eta \leq H_*(\overline{y}, \overline{t}, 0, Y) - H^*(\overline{x}, \overline{t}, 0, X) \leq H_*(\overline{y}, \overline{t}, 0, 0) - H^*(\overline{x}, \overline{t}, 0, 0) = 0$, which is a contradiction.

Therefore, sending $\rho \to 0$ and setting $K = C_H$, necessarily we have $\overline{t} = 0$. We get for all $x, y \in \mathbb{R}^N$, $t \in [0,T]$

$$\begin{aligned}&u(x,t) - u(y,t) - \eta t\\ &\leq u_0(\overline{x}) - u_0(\overline{y}) - \frac{|\overline{x} - \overline{y}|^4}{\varepsilon^4} + \frac{e^{Kt}|x-y|^4}{\varepsilon^4}\\ &\leq \|Du_0\|_\infty|\overline{x} - \overline{y}| - \frac{|\overline{x} - \overline{y}|^4}{\varepsilon^4} + \frac{e^{Kt}|x-y|^4}{\varepsilon^4}\\ &\leq \frac{3}{4^{4/3}}\|Du_0\|_\infty^{4/3}\varepsilon^{4/3} + \frac{e^{Kt}|x-y|^4}{\varepsilon^4}.\end{aligned}$$

We have used the Young inequality in the third inequality. Setting

$$\varepsilon^4 = \Big(\frac{4^{4/3}e^{Kt}|x-y|^4}{C\|Du_0\|_\infty^{3/4}}\Big)^{3/4},$$

we get

$$u(x,t) - u(y,t) - \eta t \leq \|Du_0\|_\infty e^{Kt/4}|x-y|.$$

Sending $\eta \to 0$, we have

$$u(x,t) - u(y,t) \leq \|Du_0\|_\infty e^{Kt/4}|x-y|.$$

On the one hand, by taking $x = y$ in the above inequality, we get $u \leq v$ and obtain the uniqueness of the solution. On the other hand, by choosing $u = v$, we obtain (32).

We now prove (33). Set $R := 2R_T \vee \|Du_0\|_{L^\infty(\mathbb{R})}e^{KT}$, where K is the constant given by Proposition 3. Recalling (31), in view of Lemma 9.1 in [5], there exists a constant $\tilde{L} > 0$ such that

$$|u(x,t) - u(x,s)| \leq \tilde{L}|t-s|^{1/2}$$

for all $x \in B(0, R_T)$, $s \in [t, T]$. Noting that $u(x,t) \equiv -1$ on $(\mathbb{R}^N \setminus B(0, R_T)) \times [0, T]$, we get

$$|u(x,t) - u(x,s)| \leq \tilde{L}|t-s|^{1/2}$$

for all $x \in \mathbb{R}^N$, $t, s \in [0, T]$. □

Lemma 17. *Assume* (I2)*. Define the nondecreasing function* $\Psi \in C(\mathbb{R})$ *by*

$$\Psi(r) := \begin{cases} -1 & \textit{if} \quad r \leq -\frac{3\delta_0}{4}, \\ \frac{2(2-\delta_0)}{\delta_0}(r + \frac{\delta_0}{2}) - \frac{\delta_0}{2} & \textit{if} \quad -\frac{3\delta_0}{4} \leq r \leq -\frac{\delta_0}{2}, \\ r & \textit{if} \quad -\frac{\delta_0}{2} \leq r \leq \frac{\delta_0}{2}, \\ \frac{\delta_0}{2} & \textit{if} \quad \frac{\delta_0}{2} \leq r. \end{cases}$$

There exists a constant $\overline{\lambda} \in (0, \lambda_0]$ *which depends only on* δ_0, η_0, $\|Du_0\|_\infty$ *and* $\|\nu\|_\infty$ *and satisfies*

$$u_0(\psi_\lambda(x)) \geq \Psi(u_0(x) + \lambda\eta_0) \quad \textit{for any } x \in \mathbb{R}^N, \lambda \in [0, \overline{\lambda}]. \tag{67}$$

Proof. It is easy to see inequality (67) if $|u_0(x)| \leq \delta_0$ in view of (I2). In the case where $u_0(x) > \delta_0$, we have

$$u_0(x + \lambda\nu(x)) \geq u_0(x) - \lambda\|Du_0\|_\infty|\nu(x)| \geq \delta_0 - \lambda\|Du_0\|_\infty\|\nu\|_\infty,$$

which implies inequality (67) if $\lambda \in [0, \overline{\lambda}]$ with $\overline{\lambda}\|Du_0\|_\infty\|\nu\|_\infty \leq \delta_0/2$. Finally, we consider the case where $u_0(x) < -\delta_0$. By replacing $\overline{\lambda}$ by a smaller constant if necessary, we may assume that $\lambda\eta_0 \leq \delta_0/4$ for all $\lambda \in [0, \overline{\lambda}]$. Then we have $\Psi(u_0(x) + \lambda\eta_0) = -1$, which yields a conclusion. □

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